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Does Competition Solve the Hold-up Problem?

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In an environment in which heterogeneous buyers and sellers undertake *ex ante* investments, the presence of market competition for matches provides incentives for investment but may leave inefficiencies, namely hold-up and coordination problems. This paper shows, using an explicitly non-cooperative model, that when matching is assortative and investments precede market competition, buyers' investments are constrained efficient while sellers marginally underinvest with respect to what would be constrained efficient. However, the overall extent of this inefficiency may be large. Multiple equilibria may arise; one equilibrium is characterized by efficient matches, but there can be additional equilibria with coordination failures.

Introduction

A central concern is the extent to which competitive market systems are efficient, and in the idealized model of Arrow and Debreu, efficiency follows under mild conditions, notably the absence of externalities. But in recent years, economists have become interested in studying less idealized market situations and in examining the pervasive inefficiencies that may exist. This paper studies a market situation that arises through an explicit non-cooperative game, played by buyers and sellers, where investments that determine the character of goods are chosen before market interaction occurs. Two potential inefficiencies arise: these are often referred to as the hold-up problem and coordination failures. An important part of our analysis will be to examine the connection between, as well as the extent of, the inefficiencies induced by these two problems, and whether market competition may solve them.

The hold-up problem applies when a group of agents, for example, a buyer and a seller, share some surplus from interaction and an agent making an investment is unable to receive all the benefits that accrue from that investment. The existence of the problem is generally traced to incomplete contracts: with complete contracts, the inefficiency induced by the failure to capture benefits will not persist (Grossman and Hart 1986; Grout 1984; Hart and Moore 1988; Williamson 1985). Coordination failures arise when a group of agents can realize a mutual gain only by a change in behaviour of each member of the group. For instance, a buyer may receive the marginal benefits from an investment when she is matched with a particular seller, so there is no hold-up problem, but she may be inefficiently matched with a seller; the incentive to change the match may not exist because gains may be realized only if the buyer to be displaced is willing to alter her investment.

What happens if the interaction of agents is through the marketplace? In an Arrow–Debreu competitive model, complete markets, with price-taking in each market, are assumed; if an agent chooses investment *ex ante*, then every different level of investment may be thought of as providing the agent with a different good to bring to the market (Makowski and Ostroy 1995). If a buyer wishes to choose some investment level and the seller with whom he trades prefers to trade with this buyer rather than with another buyer, then total surplus to be divided must be maximized: investment will be efficiently chosen, and there is no hold-up problem. The existence of complete markets implies that prices for all investment levels are known: complete markets imply complete contracts. In addition, as long as there are no externalities, the return from any match is independent of the actions of agents who are not part of the match, so coordination failures do not arise. However, if the marketplace is such that there is pricing only of trades that take place *ex post*, only a limited number of contracts are specified: incomplete markets imply incomplete contracts.

There are a variety of applications where understanding the effect of competition on hold-up and coordination failures is likely to be relevant. One example is a labour market where employees and employers have to make specific human capital and technological investments in advance of the matching process that leads to a successful employment relationship. Another example is the relationship between suppliers and manufacturers (see Calzolari *et al*. (2015) for an analysis of this relationship in the German car manufacturing industry). If the technology requires firm- or model-specific intermediate parts, then the absence of a long-term contract may lead to a hold-up problem with underinvestment on the part of the supplier, but competition among a, possibly small, number of suppliers may reduce the inefficiencies associated with such a hold-up problem at the cost of introducing inefficiencies that take the form of coordination failures (inefficient selection of supplier).

This paper investigates the efficiency of investments when the trading pattern and terms of trade are determined explicitly by a non-cooperative model of competition between buyers and sellers. To ensure that there are no market power inefficiencies, a model of Bertrand competition is analysed where agents invest prior to trade. There is a finite number of agents to ensure that patterns of trade can be changed by individual agents. By definition, buyers bid to trade with sellers. Contracts are the result of competition, and our interest is the degree to which hold-up and coordination problems are mitigated by competitive contracts. In this regard, it should be said that Bertrand competition in contingent contracts is ruled out; in our analysis, contracts take the form of an agreement between a buyer and a seller to trade at a particular price. We are thus investigating the efficiency of a simple trading structure rather than attempting explicitly to devise contracts to address particular problems (Aghion *et al*. 1994; Maskin and Tirole 1999; Segal and Whinston 2002).

We restrict attention to markets where the Bertrand competitive outcome is robust to the way that markets are made to clear. To be specific, we assume that buyers and sellers can be ordered by their ability to generate surplus with a complementarity between buyers and sellers. Under a weak specification of the market clearing process, this gives rise to assortative matching in the qualities of buyers and sellers, where quality is in part determined by investment choices. If investment levels are not subject to choice, then the Bertrand equilibrium is always efficient.

Consider first the sellers' equilibrium investments. We show that these investments are inefficient and a hold-up problem arises. In essence, a seller chooses investments to maximize the surplus that would be created if he were to be matched with the runner-up in the bidding to be matched with him.

We then demonstrate that buyers' investment levels are constrained efficient. For a given equilibrium match, if a buyer bids just enough to win the right to trade with a seller, then as a result of any extra investment, she would need to make only the same bid to win the right to trade with the same seller—she would receive all the marginal benefits of investment. This result is extended to show that buyers also receive the marginal value of their investments even when this involves a change in match. A consequence of this is the existence of an equilibrium outcome where all buyers make constrained efficient choices; the constraint that qualifies this equilibrium is the set of other agents' investment choices.

Compatible with constrained efficiency is an outcome where a buyer overinvests because she is matched with a seller of too high a quality because another buyer has underinvested because she, in turn, is matched with a seller of too low a quality, and vice versa. Thus coordination failures may arise with resulting inefficiency. However, we show that these inefficiencies will not arise if the returns from investments differ sufficiently across buyers.

Under concavity restrictions on the match technology, the blunted incentive faced by sellers is small, and the total cost of the inefficiency is bounded by the inefficiency that could be created by a single seller underinvesting with all others investing efficiently. However, if there are more buyers than sellers, as we assume, then the runner-up buyer to the lowest-quality seller will not be matched in equilibrium and will choose not to invest. With strong complementarities between buyers and sellers, the lowest-quality seller will not invest, and this gives the incentive to the buyer with whom he is matched, the potential runner-up in the bid for the second-lowest-quality seller, not to invest. This gives the incentive not to invest to the second-lowest seller, and so on. Thus there will be a cascade of no investment, which ensures an equilibrium far from efficiency.

However, the hold-up misincentives just described also work to reduce coordination failure inefficiencies. Sellers who change their investments and their match partner do not necessarily alter the runner-up in the bid to be matched with them. In particular, when market trading is structured so that competition among buyers is most intense no coordination problems arise on the sellers' side of the market. It is the blunted incentives created by the hold-up problem that remove the inefficiencies that come from coordination failures.

The structure of the paper is as follows. After a discussion of related literature in the next section, Section II lays down the basic model and the extensive form of the Bertrand competition game between buyers and sellers. It is then shown in Section III that with fixed investments, the competition game gives rise to an efficient outcome—buyers and sellers match efficiently. Section IV characterizes the sellers' optimal choice of *ex ante* investments for given buyers' qualities. We show that in equilibrium, sellers underinvest. We then consider in Section V the optimal choice of the buyers' *ex ante* investments. Section VI presents the equilibrium characterization. There always exists an equilibrium with efficient matches. However, depending on parameters, we show that equilibria with coordination failures may arise that lead to inefficient matches. Section VII provides concluding remarks. For ease of exposition, all proofs are relegated to the Appendix.

I. Related Literature

There is a considerable literature that analyses *ex ante* investments in a matching environment. Some of the existing papers focus on general as opposed to match-specific investments, and identify the structure of contracts (MacLeod and Malcomson 1993) or the structure of competition (Holmström 1999) and market structure (Acemoglu and Shimer 1999; Spulber 2002) that may lead to inefficiency. Other papers (Acemoglu 1997; Ramey and Watson 2001) focus on the inefficiencies induced by the probability of match break-up.[1] Kranton and Minehart (2001) consider investments in the market structure itself; specifically, markets are limited by networks that agents create through investment. A recent paper by Mailath *et al.* (2013) looks at the structure of market clearing in a very different market to ours; however, they highlight the possibility of inefficiencies due to coordination failures that can arise in their framework.

Burdett and Coles (2001), Peters and Siow (2002) and Peters (2007) focus on the efficiency of investments in a model of non-transferable utility, in other words a marriage market. The recent paper by Peters can be viewed as the non-transferable utility analogue of the present paper. With non-transferable utility, the full role of competition cannot be addressed.

The other two papers closest to our analysis are Cole *et al*. (2001a, b). They analyse a model where there are two sides of the market and match-specific investments are chosen *ex ante*. However, the matching process is modelled as a cooperative assignment game. In Cole *et al*. (2001a), there is a finite number of different types of individual on each side of the market. Efficiency can result when a condition termed double-overlapping, which requires the presence of other agents with the same characteristics as any one agent, is satisfied. Their other paper, Cole *et al*. (2001b), deals with a continuum of types; this makes it less like the setup of the present paper.

Finally, de Meza and Lockwood (2004) and Chatterjee and Chiu (2013) also analyse a matching environment with transferable utility in which both sides of the market can undertake match-specific investments. They focus on a setup that delivers inefficient investments, and explore how asset ownership may enhance welfare (as in Grossman and Hart 1986).

II. The Framework

We consider a simple matching model: S buyers match with T sellers, and we assume that the number of buyers is higher than the number of sellers ($S > T$). Each seller is assumed to match with only one buyer. Buyers and sellers are labelled, respectively, $s = 1, \ldots, S$ and $t = 1, \ldots, T$. Both buyers and sellers can make (heterogeneous) investments, denoted x_s and y_t, respectively, incurring costs $C(x_s)$ and $C(y_t)$, respectively.[2] The cost function $C(\cdot)$ is twice differentiable and strictly convex, and $C(0) = 0$. The surplus of each match is then a function of the qualities of the buyer σ and the seller τ involved in the match: $v(\sigma,\tau)$. Each buyer's quality is itself a function of the buyer's innate ability, indexed by his identity s, and the buyer's specific investment x_s, namely $\sigma(s, x_s)$. In the same way, each seller's quality is a function of the seller's innate ability, indexed by her identity t, and the seller's specific investment y_t, namely $\tau(t,y_t)$.[3]

We assume that quality is a desirable attribute and that there is *complementarity* between the qualities of the buyer and the seller involved in a match. In other words, the higher the quality of the buyer and the seller, the higher the surplus generated by the match:[4] $v_1(\sigma, \tau) > 0$, $v_2(\sigma, \tau) > 0$. Further, the marginal surplus generated by a higher quality of the buyer or of the seller in the match increases with the quality of the partner: $v_{12}(\sigma, \tau) > 0$. We also assume that the quality of the buyer depends negatively on the buyer's innate ability s, i.e. $\sigma_1(s, x_s) < 0$ (so buyer $s = 1$ is the highest-ability buyer), and positively on the buyer's specific investment x_s, i.e. $\sigma_2(s, x_s) > 0$. Similarly, the quality of a seller depends negatively on the seller's innate ability t, i.e. $\tau_1(t, y_t) < 0$ (seller $t = 1$ is the highest-ability seller), and positively on the seller's investment y_t, i.e. $\tau_2(t, y_t) > 0$. Finally, we assume that the qualities of both the buyers and the sellers satisfy a *single crossing condition* requiring that the marginal productivity of both buyers' and sellers' investments decreases in their innate ability index: $\sigma_{12}(s, x_s) < 0$ and $\tau_{12}(t, y_t) < 0$.

The combination of the assumption of complementarity and the single crossing condition gives a particular meaning to the term heterogeneous investments that we used for x_s and y_t. Indeed, in our setting, the investments x_s and y_t have a use and value in matches other than (s,t); however, these values change (decrease) with the identity of the partner, implying that at least one component of this value is 'specific' to the match in question, since we consider a discrete number of buyers and sellers.

We also assume that the surplus of each match is concave in the buyers' and sellers' qualities—$v_{11}(\sigma, \tau) < 0$, $v_{11}(\sigma, \tau) < 0$—and that the qualities of both sellers and buyers exhibit decreasing marginal returns in their investments: $\sigma_{11}(\sigma, \tau) < 0$ and $\tau_{22}(\sigma, \tau) < 0$.[5]

We assume the following extensive forms of the Bertrand competition game in which the T sellers and the S buyers engage. Buyers Bertrand compete for sellers. All buyers simultaneously and independently submit bids to the T sellers. Notice that we allow buyers to submit bids to more than one seller, possibly all sellers. Each seller observes the bids that she receives and decides which offer to accept. We assume that this decision is taken in the order of seller's identities (innate abilities) $(1, \ldots, T)$. In other words, the seller labelled 1 decides first which bid to accept. This commits the buyer selected to a match with seller 1 and automatically withdraws all bids that this buyer made to the other sellers. All other sellers and buyers observe this decision, then seller 2 decides which bid to accept. This process is repeated until seller T decides which bid to accept. Notice that since $S > T$, even seller T, the last seller to decide, can choose among multiple bids.[6]

We look for the set of *cautious equilibria* of our model so as to rule out equilibria in which (unsuccessful) bids exceed buyers' valuations. The basic idea behind this equilibrium concept is that no buyer should be willing to make a bid that would leave the buyer worse off relative to the equilibrium if accepted.[7] A cautious equilibrium is equivalent to equilibrium in weakly dominant strategies. In the construction of the cautious equilibrium, we allow buyers, when submitting a bid, to state that they are prepared to bid more if this becomes necessary. We then restrict the strategy choice of each seller to be such that each seller selects bids starting with a higher-order probability on the highest bids, and allocates a lower-order probability of being selected on a bid submitted by a buyer that did not specify such a proviso.[8]

The logic behind this additional restriction derives from the observation that in the extensive form of the Bertrand game there exists an asymmetry between the timing of buyers' bids (they are all simultaneously submitted at the beginning of the Bertrand competition subgame) and the timing of each seller's choice of the bid to accept (sellers choose their most preferred bid sequentially in a given order). This implies that while in equilibrium it is possible that a seller's choice between two identical bids is uniquely determined, this is no longer true following a deviation by a buyer whose bid in equilibrium is selected at an earlier stage of the subgame. To prevent sellers from deviating when choosing among identical bids following a buyer's deviation—that possibly does not even affect the equilibrium bids submitted to the seller in question—we chose to modify the extensive form in the way described above.

III. Bertrand Competition

We now proceed to characterize the equilibria of the model described in Section II, solving it backwards. We start from the characterization of the equilibrium of the Bertrand competition subgame, taking the investments, and hence the qualities of both sellers and buyers, as given.

To simplify the analysis below, let τ_n be the quality of seller n, $n = 1, \ldots, T$, who, as described in Section II, is the nth seller to choose her most preferred bid. The vector of sellers' qualities is then $(\tau_1, \ldots, \tau_T)$.

We first show that all the equilibria of the Bertrand competition subgame exhibit *positive assortative matching*. In other words, for given investments, matches are efficient: the buyer characterized by the kth-highest-quality matches with the seller characterized by the kth highest quality.

Lemma 1. Every equilibrium of the Bertrand competition subgame is such that every pair of equilibrium matches (σ',τ_i) and (σ'',τ_j), $i,j \in \{1,\ldots,T\}$, satisfies the following property. If $\tau_i > \tau_j$, then $\sigma' > \sigma''$.

The proof of this result (in the Appendix) is a direct consequence of the complementarity assumption of buyers' and sellers' qualities. Notice that Lemma 1 does not imply that the order of sellers' qualities, which are endogenously determined by sellers' investments, coincides with the order of sellers' identities (innate abilities).

Using Lemma 1, we can now label buyers' qualities in a way that is consistent with the way sellers' qualities are labelled. Indeed, Lemma 1 defines an equilibrium relationship between the quality of each buyer and the quality of each seller. We can therefore denote by σ_n, $n = 1, \ldots, T$, the quality of the buyer that in equilibrium matches with seller τ_n. Furthermore, we denote by $\sigma_{T+1}, \ldots, \sigma_S$ the qualities of the buyers who in equilibrium are not matched with any seller, and assume that these qualities are ordered so that $\sigma_i > \sigma_{i+1}$ for all $i = T+1, \ldots, S-1$.

Consider stage t of the Bertrand competition subgame, characterized by the fact that the seller of quality τ_t chooses her most preferred bid. The buyers who are still unmatched at this stage of the subgame are the ones with qualities $\sigma_t, \sigma_{t+1}, \ldots, \sigma_S$.[9] We define the *runner-up* buyer to the seller of quality τ_t to be the buyer, among the ones with qualities $\sigma_{t+1}, \ldots, \sigma_S$, who has the highest willingness to pay for a match with seller τ_t. This willingness to pay is the difference between the surplus of the match between the runner-up buyer and the seller in question, and the payoff that the runner-up buyer obtains if he is not successful in his bid to the seller. We denote this buyer as $r(t)$, and his quality as $\sigma_{r(t)}$. Clearly, $r(t) > t$.

This definition can be used recursively so as to define the runner-up buyer to the seller who is matched in equilibrium with the runner-up buyer to the seller of quality τ_t. We denote this buyer as $r^2(t) = r(r(t))$ and his quality as $\sigma_{r^2(t)}$, $r^2(t) > r(t) > t$. In an analogous way we can then write $r^k(t) = r(r^{k-1}(t))$ for $k = 1, \ldots, \rho_t$, where $r^k(t) > r^{k-1}(t)$, $r^1(t) = r(t)$ and $\sigma_{r^{\rho_t}(t)}$ is the quality of the last buyers in the chain of runner-ups to the seller of quality τ_t.

We now have all the elements necessary to provide a characterization of the equilibrium of the Bertrand competition subgame. In particular, we first identify the runner-up buyer to every seller, and the difference equation satisfied by the equilibrium payoffs to all sellers and buyers. This is done in the following lemma.

Lemma 2. The runner-up buyer to the seller of quality τ_t, $t = 1, \ldots, T$, is the buyer of quality $\sigma_{r(t)}$ such that

$$\sigma_{r(t)} = \max\{\sigma_i \mid i = t+1, \ldots, S \text{ and } \sigma_i \leq \sigma_t\}. \tag{1}$$

Further, the equilibrium payoffs to each buyer, $\pi^B_{\sigma_t}$, and each seller, $\pi^S_{\tau_t}$, are such that for t=1, …, T,

(2) $$\pi^B_{\sigma_t} = [v(\sigma_t, \tau_t) - v(\sigma_{r(t)}, \tau_t)] + \pi^B_{\sigma_{r(t)}},$$

(3) $$\pi^S_{\tau_t} = v(\sigma_{r(t)}, \tau_t) - \pi^B_{\sigma_{r(t)}},$$

and for $i = T + 1, \ldots, S$,

(4) $$\pi^B_{\sigma_i} = 0.$$

Notice that equation (1) identifies the runner-up buyer of the seller of quality τ_t as the buyer—other than the one of quality σ_t who in equilibrium matches with seller τ_t—who has the highest quality among the buyers with quality lower than σ_t who are still unmatched at stage t of the Bertrand competition subgame. For any seller of quality τ_t it is then possible to construct a chain of runner-up buyers: each one is the runner-up buyer to the seller who, in equilibrium, is matched with the runner-up buyer who is next ahead in the chain. Equation (1) implies that for every seller, the last buyer in the chain of runner-up buyers is the buyer of quality σ_{T+1}. This is the highest quality buyer among the ones who in equilibrium do not match with any seller. In other words, every chain of runner-up buyers has at least one buyer in common.

Given that buyers Bertrand compete for sellers, each seller will be able to capture not all the match surplus but only her outside option, which is determined by the willingness to pay of the runner-up buyer to the seller. This is the difference between the surplus of the match between the runner-up buyer and the seller in question, and the payoff that the runner-up buyer obtains in equilibrium if he is not successful in his bid to the seller: the difference equation in (3). Given that the quality of the runner-up buyer is lower than the quality of the buyer with whom the seller is matched in equilibrium, the share of the surplus that each seller is able to capture does not coincide with the entire surplus of the match. The payoff to each buyer is then the difference between the surplus of the match and the runner-up buyer's bid: the difference equation in (2). The characterization of the equilibrium of the Bertrand competition subgame is summarized in the following proposition.

Proposition 1. For any given vector of sellers' qualities $(\tau_1, \ldots, \tau_T)$ and corresponding vector of buyers' qualities $(\sigma_1, \ldots, \sigma_S)$, the unique equilibrium of the Bertrand competition subgame is such that every pair of equilibrium matches (σ_i, τ_i) and (σ_j, τ_j), $i,j \in \{1, \ldots, T\}$, is such that

(5) $$\text{if } \tau_i > \tau_j, \text{then } \sigma_i > \sigma_j.$$

Further, the equilibrium shares of the match surplus received by each buyer of quality σ_t and each seller of quality τ_t, $t = 1, \ldots, T$, are such that

(6) $$\pi^B_{\sigma_t} = [v(\sigma_t, \tau_t) - v(\sigma_{r(t)}, \tau_t)] + \sum_{k=1}^{\rho_t} \Big[v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)}) \Big],$$

$$\pi^S_{\tau_t} = v(\sigma_{r(t)}, \tau_t) - \sum_{k=1}^{\rho_t} \Big[v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)}) \Big], \tag{7}$$

where $r^{\rho_t}(t) = T + 1$ and $v(\sigma_{r^{\rho_t}(t)}, \tau_{r^{\rho_t}(t)}) = v(\sigma_{r^{\rho_t+1}(t)}, \tau_{r^{\rho_t}(t)}) = 0$.

Consider the special case in which the order of sellers' qualities coincides with the order of their innate abilities. This implies that sellers select their most preferred bid in the decreasing order of their qualities: $\tau_1 > \ldots > \tau_T$. From Lemma 2—condition (1)—this also implies that the runner-up buyer to the seller of quality τ_t is the buyer of quality σ_{t+1} for $t = 1, \ldots, T$. The following corollary of Proposition 1 specifies the equilibrium of the Bertrand competition subgame in this case.

Corollary 1. For any given ordered vector of sellers' qualities $(\tau_1, \ldots, \tau_T)$ such that $\tau_1 > \ldots > \tau_T$ and corresponding vector of buyers' qualities $(\sigma_1, \ldots, \sigma_S)$, the unique equilibrium of the Bertrand competition subgame is such that the equilibrium matches are (σ_k, τ_k), $k = 1, \ldots, T$, and the shares of the match surplus received by each buyer of quality σ_t and each seller of quality τ_t are such that

$$\pi^B_{\sigma_t} = \sum_{h=t}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)], \tag{8}$$

$$\pi^S_{\tau_t} = v(\sigma_{t+1}, \tau_t) - \sum_{h=t+1}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)]. \tag{9}$$

The main difference between Proposition 1 and Corollary 1 can be described as follows. Consider the subgame in which the seller of quality τ_t chooses among her bids, and let $(\tau_1, \ldots, \tau_T)$ be an ordered vector of qualities as in Proposition 1. This implies that $\sigma_t > \sigma_{t+1} > \sigma_{t+2}$. The runner-up buyer to the seller with quality τ_t is then the buyer of quality σ_{t+1}, and the willingness to pay of this buyer (hence the share of the surplus accruing to seller τ_t) is, from (3),

$$v(\sigma_{t+1}, \tau_t) - \pi^B_{\sigma_{t+1}}. \tag{10}$$

Notice further that since the runner-up buyer to seller τ_{t+1} is σ_{t+2} from (2), the payoff to the buyer of quality σ_{t+1} is

$$\pi^B_{\sigma_{t+1}} = v(\sigma_{t+1}, \tau_{t+1}) - v(\sigma_{t+2}, \tau_{t+1}) + \pi^B_{\sigma_{t+2}}. \tag{11}$$

Substituting (11) into (10), we obtain that the willingness to pay of the runner-up buyer σ_{t+1} is then

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) - \pi^B_{\sigma_{t+2}}. \tag{12}$$

Consider now a new vector of sellers' qualities $(\tau_1, \ldots, \tau'_{t-1}, \tau_t, \tau'_{t+1}, \ldots, \tau_T)$, where the qualities τ_i for every i different from $t-1$ and $t+1$ are the same as the ones in the ordered vector $(\tau_1, \ldots, \tau_T)$. Assume that $\tau'_{t-1} = \tau_{t+1} < \tau_t$ and $\tau'_{t+1} = \tau_{t-1} > \tau_t$. This assumption implies that the vector of buyers' qualities $(\sigma'_1, \ldots, \sigma'_S)$ differs from the ordered vector of buyers' qualities $(\sigma_1, \ldots, \sigma_S)$ only in its $(t-1)$th and $(t+1)$th components that are such that $\sigma'_{t-1} = \sigma_{t+1} < \sigma_t$ and $\sigma'_{t+1} = \sigma_{t-1} > \sigma_t$. From (1), we have that the runner-up buyer for seller τ_t is now buyer σ_{t+2}, and the willingness to pay of this buyer is

$$v(\sigma_{t+2}, \tau_t) - \pi^B_{\sigma_{t+2}}. \tag{13}$$

Comparing (12) with (13), we obtain, by the complementarity assumption $v_{12}(\sigma, \tau) > 0$, that

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t).$$

In other words, the willingness to pay of the runner-up buyer to seller τ_t in the case considered in Corollary 1 is strictly greater than the willingness to pay of the runner-up buyer to seller τ_t in the special case of Proposition 1 that we just considered. The reason is that in the latter case, there is one less buyer σ_{t+1} to actively compete for the match with seller τ_t.

This comparison is generalized in the following proposition

Proposition 2. Let $(\tau_1, \ldots, \tau_T)$ be an ordered vector of sellers' qualities so that $\tau_1 > \ldots > \tau_T$, and let $(\tau'_1, \ldots, \tau'_T)$ be any permutation of the vector $(\tau_1, \ldots, \tau_T)$ with the same tth element: $\tau'_t = \tau_t$ such that there exists an $i < t$ that permutes into a τ'_j $(\tau_i = \tau'_j)$ with $j > t$. Denote by $(\sigma_1, \ldots, \sigma_T)$ and $(\sigma'_1, \ldots, \sigma'_T)$ the corresponding vectors of buyers' qualities. Then the payoff of seller τ_t, as in (9), is greater than the payoff of seller τ'_t, as in (7):

$$\begin{aligned} & v(\sigma_{t+1}, \tau_t) - \sum_{h=t+1}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \\ & > v(\sigma'_{r(t)}, \tau'_t) - \sum_{k=1}^{\rho'_t} \left[v(\sigma'_{r^k(t)}, \tau'_{r^k(t)}) - v(\sigma'_{r^{k+1}(t)}, \tau'_{r^k(t)}) \right]. \end{aligned} \tag{14}$$

Proposition 2 allows us to conclude that when sellers select their preferred bid in the decreasing order of their qualities, competition among buyers for each match is at its peak.[10] This is apparent when we consider the case where the order in which sellers select their most preferred bid is the increasing order of their qualities: $\tau_1 < \ldots < \tau_T$. In this case, according to (1), the runner-up buyer to each seller has quality σ_{T+1}. This implies that the payoff to each seller $t=1, \ldots, T$ is

$$\pi^S_{\tau_t} = v(\sigma_{T+1}, \tau_t).$$

In this case only two buyers—the buyer of quality σ_t and the buyer of quality σ_{T+1}—actively compete for the match with seller τ_t, and sellers' payoffs are at their minimum.

We assume that sellers choose their most preferred bid in the decreasing order of their innate ability. Notice that this does not necessarily mean that sellers choose their most preferred bid in the decreasing order of their qualities $\tau_1 > \ldots > \tau_T$, hence competition among buyers is at its peak. Indeed, sellers' qualities are endogenously determined in what follows.

We conclude this section by observing that from Proposition 1, the buyer's equilibrium payoff $\pi^B_{\sigma_t}$ is the sum of the social surplus produced by the equilibrium match $v(\sigma_t, \tau_t)$ and an expression $\mathcal{B}_{\sigma_t}$ that does not depend on the quality σ_t of the buyer involved in the match. In particular, this implies that $\mathcal{B}_{\sigma_t}$ does not depend on the match-specific investment of the buyer of quality σ_t:

$$\pi^B_{\sigma_t} = v(\sigma_t, \tau_t) + \mathcal{B}_{\sigma_t}. \tag{15}$$

Moreover, from (7), each seller's equilibrium payoff $\pi^S_{\tau_t}$ is also the sum of the surplus generated by the inefficient (if it occurs) match of the seller of quality τ_t with the runner-up buyer of quality $\sigma_{r(t)}$, and an expression $\mathcal{S}_{\tau_t}$ that does not depend on the investment of the seller of quality τ_t:

$$\pi^S_{\tau_t} = v(\sigma_{r(t)}, \tau_t) + \mathcal{S}_{\tau_t}. \tag{16}$$

Of course, when sellers select their bids in the decreasing order of their qualities, the runner-up buyer to seller t is the buyer of quality σ_{t+1}, from (1). Therefore equation (16) becomes

$$\pi^S_{\tau_t} = v(\sigma_{t+1}, \tau_t) + \mathcal{S}_{\tau_t}. \tag{17}$$

These conditions play a crucial role when we analyse the efficiency of the investment choices of both buyers and sellers.

IV. Sellers' Investments

We now move back one period and consider the buyers' and sellers' simultaneous-move investment game.

In this section we derive the sellers' best reply and we provide a partial characterization of the equilibrium in which we focus exclusively on the sellers' investment choices. We therefore take the qualities of buyers as given by the ordered vector $(\sigma_{(1)}, \ldots, \sigma_{(S)})$, and determine the sellers' *ex ante* optimal investment choices given their identities.

Notice that in characterizing the sellers' investment choices we cannot bluntly apply Corollary 1 as the characterization of the equilibrium of the Bertrand competition subgame. Indeed, the order in which sellers choose among bids in this subgame is determined by the sellers' innate abilities rather than by their qualities. This implies that unless sellers' qualities (which are endogenously determined) have the same order as sellers' innate abilities, it is possible that sellers do not choose among bids in the decreasing order of their marginal contribution to a match (at least off the equilibrium path).

For a given level of buyer's investment x_s, denote by $y(t,s)$ the efficient investment of seller t when matched with the buyer of quality $\sigma_{(s)}$ defined as

(18) $$y(t,s) = \text{argmax}_y\ v(\sigma_{(s)}, \tau(t,y)) - C(y).$$

We can now state the following property of the sellers' investment game

Proposition 3. In every equilibrium of the investment game, the sellers' optimal choices of investments are such that seller t chooses investment $y(t,t{+}1)$, as defined in (18).

Proposition 3 implies two different features of the sellers' optimal investment choice. First, the sellers under-invest. The nature of the Bertrand competition game is such that each seller is able to capture not all the match surplus but only the outside option that is determined by the willingness to pay of the runner-up buyer for the match. Since the match between a seller and her runner-up buyer yields a match surplus that is strictly lower than the equilibrium surplus produced by the same seller, the share of the surplus that the seller is able to capture does not coincide with the entire surplus of the match.

Corollary 2. Each seller $t = 1, \ldots, T$ chooses an inefficient investment level $y(t,t + 1)$. The investment $y(t,t + 1)$ is strictly lower than the investment $y(t,t)$ that it would be efficient for seller t to choose, given the equilibrium match of buyer t with seller t.

Second, the order of the sellers' qualities $\tau(t,y(t,t + 1))$ coincides with the order of the sellers' innate abilities t. Two features of the sellers' investment decision explain this result. First, each seller's payoff is completely determined by the seller's outside option and hence is independent of the identity and quality of the buyer with whom he is matched. Second, sellers choose their bids in the decreasing order of their innate abilities, and this order is independent of sellers' investments. These two features of the model, together with positive assortative matching (Lemma 1), imply that when a seller chooses an investment that yields a quality higher than the one with higher innate ability, it modifies the set of unmatched buyers, and hence of bids from among which the seller chooses, only by changing the bid of the buyer with whom the seller will be matched in equilibrium. Hence this change will not affect the outside option and payoff of this seller, implying that the optimal investment cannot exceed the optimal investment of the seller with higher innate ability. Therefore sellers have no incentive to modify the order of their innate ability at an *ex ante* stage.

V. Buyers' Investments

In this section we derive the buyers' optimal investments. We take the quality of sellers $\tau_1 > \ldots > \tau_T$ to be given and, from Proposition 3, to coincide with the order of the sellers' innate ability, and derive the buyers' optimal choice of investment given their own identity (innate ability). Corollary 1 provides the characterization of the unique equilibrium of the Bertrand competition subgame in this case.

In Section VI, we first show that it is possible to construct buyers' investments that lead to an efficient equilibrium of the investment game: the order of the induced qualities $\sigma(s,x_s)$, $s = 1, \ldots, S$, coincides with the order of the buyers' identities s, $s = 1, \ldots, S$. We then show that it is possible to construct buyers' investments that lead to inefficient equilibria, such that the order of the buyers' identities differs from the order of their induced qualities.

Notice that each buyer's investment choice is constrained efficient given the equilibrium match and the quality of the seller with whom the buyer is matched. Indeed, the Bertrand competition game will make each buyer residual claimant of the surplus produced in his equilibrium match. Therefore the buyer is able to appropriate the marginal returns from his investment, thus his investment choice is constrained efficient given the equilibrium match.

Assume that the equilibrium match is the one between buyer s and seller t. From equation (15), the optimal investment choice $x_s(t)$ for buyer s is the solution to the problem

$$(19) \qquad x_s(t) = \operatorname{argmax}_x \ \pi^B_{\sigma(s,x)} - C(x) = v(\sigma(s,x),\tau_t) - \mathcal{B}_{\sigma(s,x)} - C(x).$$

This investment choice is defined by the following necessary and sufficient first-order conditions of problem (19):

$$(20) \qquad v_1(\sigma(s,x_s(t)),\tau_t)\ \sigma_2(s,x_s(t)) = C'(x_s(t)),$$

where $C'(\cdot)$ is the first derivative of the cost function $C(\cdot)$.

Notice that (20) follows from the fact that $\mathcal{B}_{\sigma(s,x)}$ does not depend on the quality $\sigma(s, x)$ of buyer s, and hence on the match-specific investment x of buyer s.

The following result characterizes the properties of the investment choice $x_s(t)$ of buyer s, and his quality $\sigma(s,x_s(t))$.

Proposition 4. For any given equilibrium match $(\sigma(s,x_s(t)),\tau_t)$, the investment choice $x_s(t)$ of buyer s, as defined in (20), is constrained efficient.

Furthermore, the optimally chosen quality $\sigma(s,x_s(t))$ of buyer s decreases in both the buyer's identity s and the seller's identity t:

$$\frac{\mathrm{d}\sigma(s,x_s(t))}{\mathrm{d}s} < 0, \quad \frac{\mathrm{d}\sigma(s,x_s(t))}{\mathrm{d}t} < 0.$$

VI. Equilibria

In this section we characterize the set of equilibria of the investment game. We first define an equilibrium of this game. Let $(s_1, \ldots, s_S)$ denote a permutation of the vector of buyers' identities $(1, \ldots, S)$. An equilibrium of the investment game is a set of sellers' optimal investment choices $y(t,t+1)$ as in Proposition 3, and a set of buyers' optimal investment choices $x_{s_i}(i)$ as defined in (20), such that the resulting buyers' qualities have the same order as the identity of the associated sellers:

$$(21) \qquad \sigma(s_i, x_{s_i}(i)) = \sigma_i < \sigma(s_{i-1}, x_{s_{i-1}}(i-1)) = \sigma_{i-1} \quad \text{for all } i = 2,\ldots,S,$$

where σ_i denotes the ith element of the equilibrium ordered vector of qualities $(\sigma_1, \ldots, \sigma_S)$.[11]

Notice that this equilibrium definition allows for the order of buyers' identities to differ from the order of their qualities and therefore from the order of the identities of the sellers with whom each buyer is matched.

We proceed to show the existence of an efficient equilibrium of our model. This is the equilibrium of the investment game such that the order of buyers' qualities coincides with the order of buyers' identities. From Lemma 1, the efficient equilibrium matches are $(\sigma(t, x_t(t)), \tau_t)$, $t = 1, \ldots, T$.

Proposition 5. The equilibrium of the buyers' investment game characterized by $s_i = i$, $i = 1, \ldots, S$, always exists and is efficient.

The intuitive argument behind this result is simple to describe. The payoff to buyer i, namely $\pi_i^B(\sigma) - C(x(i, \sigma))$, changes as buyer i matches with a higher-quality seller, brought about by increased investment.[12] However, the payoff is continuous at any point, such as σ_{i-1}, where in the continuation Bertrand game the buyer matches with a different seller.[13] If the equilibrium considered is the efficient one—$s_i = i$ for i=1, …, S—then the payoff to buyer i is monotonic decreasing in any interval to the right of (σ_{i+1}, σ_{i-1}) and increasing in any interval to the left. Therefore this payoff has a unique global maximum. Hence buyer i has no incentive to deviate and change his investment choice.

If instead we consider an inefficient equilibrium—an equilibrium where $s_1, \ldots, s_S$ differs from 1, …, S—then the payoff to buyer i is still continuous at any point, such as $\sigma(s_i, x_{s_i}(i))$, in which in the continuation Bertrand game the buyer gets matched with a different seller. However, this payoff is no longer monotonic decreasing in any interval to the right of $(\sigma(s_{i+1}, x_{s_{i+1}}(i+1)), \sigma(s_{i-1}, x_{s_{i-1}}(i-1)))$ and increasing in any interval to the left. In particular, this payoff is increasing at least in the right neighbourhood of the switching points $\sigma(s_h, x_{s_h}(h))$ for $h = 1, \ldots, i-1$, and decreasing in the left neighbourhood of the switching points $\sigma(s_k, x_{s_k}(k))$ for $k = i+1, \ldots, N$.

This implies that depending on the values of parameters, these inefficient equilibria may or may not exist. We show below that it is possible to construct inefficient equilibria if two buyers' qualities are close enough. Alternatively, for given buyers' qualities, inefficient equilibria do not exist if the sellers' qualities are close enough.

Proposition 6. Given any vector of sellers' quality functions ($\tau(1,\cdot)$, …, $\tau(T,\cdot)$), it is possible to construct an inefficient equilibrium of the buyers' investment game such that there exists at least an i that satisfies $s_i < s_{i-1}$. Moreover, given any vector of buyers' quality functions ($\sigma(s_1,\cdot)$, …, $\sigma(s_S,\cdot)$), it is possible to construct an ordered vector of sellers' quality functions ($\tau(1,\cdot)$, …, $\tau(T,\cdot)$) such that there does not exist any inefficient equilibrium of the buyers' investment game.

The intuition of why such result holds is simple to highlight. The continuity of each buyer's payoff implies that when two buyers have similar innate abilities, just as it is not optimal for each buyer to deviate when he is matched efficiently, it is also not optimal for him to deviate when he is inefficiently assigned to a match. Indeed, the differences in buyers' qualities are almost entirely determined by the differences in the qualities of the sellers with whom they are matched rather than by the differences in buyers' innate abilities. This implies that when a buyer of low ability has undertaken a high investment with the purpose of being matched with a better seller, it is not worth the buyer of immediately higher ability trying to outbid him. The willingness to pay of the lower-ability buyer for the match with the better seller is in fact enhanced by this higher investment. Therefore the gains from outbidding this buyer do not justify the high investment of the higher-ability buyer. Indeed, in the Bertrand competition game, each buyer is able to capture just the difference between the match surplus and the willingness to pay for the match of the runner-up buyer, who would be, in this outbidding attempt, the low-ability buyer who undertook the high investment.

Conversely, if sellers' qualities are similar, then the differences in buyers' qualities are almost entirely determined by the differences in buyers' innate abilities, implying that it is not possible to construct an inefficient equilibrium of the buyers' investment game. In this case, the improvement in a buyer's incentives to invest due to a matching with a better seller are more than compensated by the decrease in the buyer's incentives induced by the lower innate ability of the buyer.

We conclude that buyers' investments are constrained efficient while sellers underinvest. It might seem at first sight that an envelope condition would ensure that the inefficiency associated with any seller's investment choice is small. Under concavity restrictions, we would expect the marginal decision of the seller to lead to less inefficiency than if it had been the decision of any other seller. This argument suggests the result that the extent of total underinvestment inefficiency in the market is bounded by what could be created from one seller (the most efficient one) choosing the level of investment appropriate for a match with the best unmatched buyer.[14] However, the complementarities that exist between buyers and sellers could still lead to the inefficiency created by a single seller being large. The lowest-quality seller chooses an investment that would have been efficient if he had been matched with the buyer who is unmatched; this buyer will choose not to invest. The complementarity effect may be strong enough to ensure that the seller would choose zero investment. This in turn will lead the buyer who is matched with this seller to also choose zero investment. This gives zero investment incentives to the second-lowest seller, and so on. It is then possible to construct an equilibrium where no investment occurs and inefficiencies are maximized.

VII. Concluding Remarks

When buyers and sellers can undertake heterogeneous investments, Bertrand competition for matches yields a number of inefficiencies. In particular, sellers underinvest but select efficient matches. The interaction of buyers and sellers can lead to the aggregate extent of this inefficiency being large. Buyers choose constrained efficient investments, but it is possible to construct equilibria in which buyers end up in inefficient matches: the order of the buyers' induced qualities differs from the order of their innate abilities.

Understanding the implications of competition for the hold-up problem and coordination failures helps in identifying the inefficiencies present in the concrete applications mentioned above. For example, it might clarify why the relationship between suppliers and manufacturers in the German car manufacturing industry is characterized not only by a level of competition among a possibly small number of suppliers for each innovative part, but also by the presence of long-term relational contracts among suppliers and manufacturers that reduce the inefficiency identified in our analysis.

One assumption is critical in our analysis. Sellers choose their most preferred bid in the order of their innate ability. In Felli and Roberts (2001), we analyse the effect of this assumption in two models: one where only sellers undertake *ex ante* investment, and one where only buyers undertake *ex ante* investment.

In these models, we characterize the equilibria when sellers select their most preferred bid in an arbitrary order. We show that competition among buyers is not as intense as in the model analysed here, leading to a higher underinvestment on the part of the sellers as well as to the possibility that equilibrium matches are inefficient on the sellers' side: the order of the sellers' induced qualities may differ from the order of their innate abilities. We then endogenize the order in which sellers select their match by letting sellers bid for their position in the queue. We show that in this case the equilibrium order will coincide with the decreasing order of the sellers' innate abilities, the one analysed above.

The extensive form of our matching game plays a critical role. One could envisage a double auction model where both buyers and sellers make bids. Depending on the particular equilibrium that results, the different inefficiencies that we have highlighted above will be shared by both sides of the market, with underinvestments and coordination failures being a feature of the equilibrium investments of buyers and sellers.

APPENDIX

Proof of Lemma 1 Assume by way of contradiction that equilibrium matches are not assortative: there exists a pair of equilibrium matches (σ'',τ_i) and (σ',τ_j) such that $\tau_i > \tau_j$ and $\sigma' > \sigma''$. Denote by $b(\tau_i)$, respectively $b(\tau_j)$, the bids accepted in equilibrium by the seller of quality τ_i, respectively of quality τ_j.

Consider first the match (σ'',τ_i). For this match to occur in equilibrium, we need that it is not optimal for the buyer of quality σ'' to match with the seller of quality τ_j rather than τ_i. If buyer σ'' deviates and does not submit a bid that will be selected by seller τ_i, then two situations may occur, depending on whether the seller of quality τ_i chooses her bid before $(i < j)$ or after $(i > j)$ the seller of quality τ_j. In particular, if τ_i chooses her bid before τ_j, then following the deviation of the buyer of quality σ'', a different buyer will be matched with seller τ_i. Then the competition for the seller of quality τ_{i+1} will be won either by the same buyer as in the absence of the deviation or, if that buyer has already been matched, by another buyer who now would not be bidding for subsequent sellers.

Repeating this argument for subsequent sellers, we conclude that when following a deviation by buyer σ'' it is the turn of the seller of quality τ_j to choose her most preferred bid, the set of unmatched buyers, excluding buyer σ'', is depleted of exactly one buyer, compared with the set of unmatched buyers when in equilibrium the seller of quality τ_j chooses her most preferred bid. Hence the maximum bids of these buyers $\widehat{b}(\tau_j)$ cannot be higher than the equilibrium bid $b(\tau_j)$ of the buyer of quality σ': $\widehat{b}(\tau_j) \leq b(\tau_j)$.[15]

Therefore for (σ'',τ_i) to be an equilibrium match, we need that

$$v(\sigma'',\tau_i) - b(\tau_i) \geq v(\sigma'',\tau_j) - \widehat{b}(\tau_j),$$

or given that, as argued above, $\widehat{b}(\tau_j) \leq b(\tau_j)$, we need that the following necessary condition is satisfied:

$$\text{(A1)} \qquad v(\sigma'',\tau_i) - b(\tau_i) \geq v(\sigma'',\tau_j) - b(\tau_j).$$

Alternatively, if τ_i chooses her bid after τ_j, then for (σ'',τ_i) to be an equilibrium match, we need that buyer σ'' does not find it optimal to deviate and outbid the buyer of quality σ' by submitting bid $b(\tau_j)$. This equilibrium condition therefore coincides with (A1).

Consider now the equilibrium match (σ',τ_j). For this match to occur in equilibrium, we need that the buyer of quality σ' does not want to deviate and be matched with the seller of quality τ_i rather than τ_j. As discussed above, depending on whether the seller of quality τ_j chooses her bid before ($j < i$) or after ($j > i$) the seller of quality τ_i, the following is a necessary condition for (σ', τ_j) to be an equilibrium match:

$$\text{(A2)} \qquad v(\sigma', \tau_j) - b(\tau_j) \geq v(\sigma', \tau_i) - b(\tau_i).$$

The inequalities (A1) and (A2) imply that

$$\text{(A3)} \qquad v(\sigma'', \tau_i) + v(\sigma', \tau_j) \geq v(\sigma', \tau_i) + v(\sigma'', \tau_j).$$

Condition (A3) contradicts the complementarity assumption $v_{12}(\sigma,\tau) > 0$.

Proof of Lemma 2 Assume that all sellers and all buyers have different induced quality. We proceed by induction on the number of sellers still to be matched. Without any loss of generality, take $S = T+1$. Consider the (last) stage T of the Bertrand competition game. In this stage, only two buyers are unmatched, and from Lemma 1 they have qualities σ_T and σ_{T+1}. Clearly the only possible runner-up to seller T is the buyer of quality σ_{T+1}, and given that, by Lemma 1, $\sigma_T > \sigma_{T+1}$, the quality of this buyer satisfies (1).

Let $b(\sigma_T)$ and $b(\sigma_{T+1})$ denote the bids submitted to seller T by the two buyers with qualities σ_T and σ_{T+1}. Seller T clearly chooses the highest of these two bids.

The buyer of quality σ_{T+1} generates surplus $v(\sigma_{T+1},\tau_T)$ if selected by seller T, while the buyer of quality σ_T generates surplus $v(\sigma_T,\tau_T)$ if selected. Hence $v(\sigma_{T+1},\tau_T)$ is the maximum willingness to bid of the runner-up buyer σ_{T+1}, while $v(\sigma_T,\tau_T)$ is the maximum willingness to bid of the buyer of quality σ_T. Notice that from $\sigma_T > \sigma_{T+1}$ and $v_1(\sigma,\tau) > 0$, we have $v(\sigma_T, \tau_T) > v(\sigma_{T+1},\tau_T)$. Buyer σ_T therefore submits a bid equal to the minimum necessary to outbid buyer σ_{T+1}. Buyer σ_{T+1}, for his part, has an incentive to deviate and outbid buyer σ_T for any bid $b(\sigma_T) < v(\sigma_{T+1},\tau_T)$. Therefore the unique equilibrium is such that both buyers' equilibrium bids are$b(\sigma_T) = b(\sigma_{T+1}) = v(\sigma_{T+1},\tau_T)$.[16]

Consider now the stage $t < T$ of the Bertrand competition game. The induction hypothesis is that the runner-up buyer for every seller of quality $\tau_{t+1}, \ldots, \tau_T$ is defined in (1). Further, the shares of surplus accruing to the sellers of qualities τ_j, $j = t+1, \ldots, T$, and to the buyers of qualities σ_j, $j = t + 1, \ldots, S$, are

$$\text{(A4)} \qquad \widehat{\pi}^B_{\sigma_j} = [v(\sigma_j, \tau_j) - v(\sigma_{r(j)}, \tau_j)] + \widehat{\pi}^B_{\sigma_{r(j)}},$$

$$\text{(A5)} \qquad \widehat{\pi}^S_{\tau_j} = v(\sigma_{r(j)}, \tau_j) - \widehat{\pi}^B_{\sigma_{r(j)}}.$$

From Lemma 1, the buyer of quality σ_t will match with the seller of quality τ_t, which implies that the runner-up buyer for seller τ_t has to be one of the buyers with qualities $\sigma_{t+1}, \ldots, \sigma_{T+1}$. Each buyer will bid an amount for every seller, which gives him the same payoff as he receives in equilibrium. To prove that the quality of the runner-up buyer satisfies (1), we need to rule out that the quality of the runner-up buyer is $\sigma_{r(t)} > \sigma_t$, and if $\sigma_{r(t)} \leq \sigma_t$, that there exists another buyer of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$.

Assume first, by way of contradiction, that $\sigma_{r(t)} > \sigma_t$. Then the willingness to pay of the runner-up buyer for the match with seller τ_t is the difference between the surplus generated by the match of the runner-up buyer of quality $\sigma_{r(t)}$ and the seller of quality τ_t minus the payoff that the buyer would get according to the induction hypothesis by moving to stage $r(t)$ of the Bertrand competition game:

$$\text{(A6)} \qquad v(\sigma_{r(t)}, \tau_t) - \widehat{\pi}^B_{\sigma_{r(t)}}.$$

From the induction hypothesis (A4), we get that the payoff $\widehat{\pi}^B_{\sigma_{r(t)}}$ is

$$\text{(A7)} \qquad \widehat{\pi}^B_{\sigma_{r(t)}} = v(\sigma_{r(t)}, \tau_{r(t)}) - v(\sigma_{r^2(t)}, \tau_{r(t)}) + \widehat{\pi}^B_{\sigma_{r^2(t)}},$$

where, from the induction hypothesis, $\sigma_{r^2(t)} < \sigma_{r(t)}$. Substituting (A7) into (A6), we get that the willingness to pay of a runner-up buyer of quality $\sigma_{r(t)}$ for the match with the seller of quality τ_t can be written as

$$\text{(A8)} \qquad v(\sigma_{r(t)}, \tau_t) - v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_{r(t)}) - \widehat{\pi}^B_{\sigma_{r^2(t)}}.$$

Consider now the willingness to pay of the buyer of quality $\sigma_{r^2(t)}$ for the match with the same seller of quality τ_t. This is

$$\text{(A9)} \qquad v(\sigma_{r^2(t)}, \tau_t) - \widehat{\pi}^B_{\sigma_{r^2(t)}}.$$

By definition of the runner-up buyer, the willingness to pay of the buyer of quality $\sigma_{r(t)}$, as in (A8), must be greater than or equal to the willingness to pay of the buyer of quality $\sigma_{r^2(t)}$, as in (A9). This inequality is satisfied if and only if

$$\text{(A10)} \qquad v(\sigma_{r(t)}, \tau_t) + v(\sigma_{r^2(t)}, \tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_t).$$

Since $\sigma_{r(t)} > \sigma_t$, from Lemma 1, $\tau_{r(t)} > \tau_t$. The latter inequality together with $\sigma_{r(t)} > \sigma_{r^2(t)}$ allows us to conclude that (A10) is a contradiction to the complementarity assumption $v_{12}(\sigma,\tau) > 0$.

Assume now, by way of contradiction, that $\sigma_{r(t)} \leq \sigma_t$ but there exists another buyer of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$. The definition of the runner-up buyer implies that his willingness to pay, as in (6), for the match with the seller of quality τ_t, is greater than the willingness to pay $v(\sigma_i, \tau_t) - \widehat{\pi}^B_{\sigma_i}$ of the buyer of quality σ_i, for the same match:

$$\text{(A11)} \qquad v(\sigma_{r(t)}, \tau_t) - \widehat{\pi}^B_{\sigma_{r(t)}} \geq v(\sigma_i, \tau_t) - \widehat{\pi}^B_{\sigma_i}.$$

Moreover, for $(\sigma_{r(t)}, \tau_{r(t)})$ to be an equilibrium match, buyer $\sigma_{r(t)}$ should have no incentive to be matched with seller τ_i instead. This implies, using an argument identical to the one presented in the proof of Lemma 1, that the following necessary condition needs to be satisfied:

$$\text{(A12)} \qquad \widehat{\pi}^B_{\sigma_{r(t)}} = v(\sigma_{r(t)}, \tau_{r(t)}) - b(\tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_i) - b(\tau_i),$$

where $b(\tau_{r(t)})$ and $b(\tau_i)$ are the equilibrium bids accepted by seller $\tau_{r(t)}$, respectively τ_i. Further, the equilibrium payoff to buyer σ_i is

$$\text{(A13)} \qquad \widehat{\pi}^B_{\sigma_i} = v(\sigma_i, \tau_i) - b(\tau_i).$$

Substituting (A12) and (A13) into (A11), we obtain that for (11) to hold, the following necessary condition needs to be satisfied:

$$\text{(A14)} \qquad v(\sigma_{r(t)}, \tau_t) + v(\sigma_i, \tau_i) \geq v(\sigma_i, \tau_t) + v(\sigma_{r(t)}, \tau_i).$$

Since by assumption $\sigma_t \geq \sigma_i$, from Lemma 1, $\tau_t > \tau_i$. The latter inequality together with $\sigma_i > \sigma_{r(t)}$ implies that (14) is a contradiction to the complementarity assumption $v_{12}(\sigma,\tau) > 0$. This concludes the proof that the quality of the runner-up buyer for seller τ_t satisfies (1).

An argument similar to the one used in the analysis of stage T of the Bertrand competition subgame concludes the proof of Lemma 2 by showing that the buyer of quality σ_t submits in equilibrium a bid equal to the willingness to pay of the runner-up buyer to seller τ_t as in (A6). This bid is the equilibrium payoff to the seller of quality τ_t, and coincides with (A3). The equilibrium payoff to the buyer of quality σ_t is then the difference between the match surplus $v(\sigma_t, \tau_t)$ and the equilibrium bid in (A6) as in (A2).

Proof of Proposition 1 Condition (5) is nothing but a restatement of Lemma 1. The proof of (6) and (7) follows directly from Lemma 2. In particular, solving recursively (2), using (4), we obtain (6); then substituting (6) into (3), we obtain (7).

Proof of Corollary 1 This result follows directly from Lemma 1, Lemma 2 and Proposition 1. In particular, (1) implies that when $(\tau_1, \ldots, \tau_T)$ and $(\sigma_1, \ldots, \sigma_S)$ are ordered vectors of qualities, $\sigma_{r(t)} = \sigma_{t+1}$ for $t = 1, \ldots, T$. Then substituting the identity of the runner-up buyer in (6) and (7), we obtain (8) and (9).

Lemma A1

Given any ordered vector of sellers' qualities $(\tau_1, \ldots, \tau_T)$ and the corresponding vector of buyers' qualities $(\sigma_1, \ldots, \sigma_S)$, for $t = 1, \ldots, T-1$ and $m = 1, \ldots, T-t$ we have

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{m}[v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+m+1}, \tau_t). \tag{A15}$$

Proof. We proceed by induction. In the case m=1, inequality (A15) becomes

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t),$$

which is satisfied by the complementarity assumption $v_{12}(\sigma,\tau) > 0$, given that $\sigma_{t+1} > \sigma_{t+2}$ and $\tau_t > \tau_{t+1}$. Assume now that for $1 \le n < m$, the following condition holds:

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{n}[v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+n+1}, \tau_t). \tag{A16}$$

We need to show that (A15) holds for m=n+1. Inequality (A15) can be written as

$$\begin{aligned} &v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{n}[v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] \\ &\quad - [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] \\ &> v(\sigma_{t+n+2}, \tau_t). \end{aligned} \tag{A17}$$

Substituting the induction hypothesis (A16) into (A17), we obtain

$$\begin{aligned} &v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{n}[v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] \\ &\quad - [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] \\ &> v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}). \end{aligned} \tag{A18}$$

Notice now that the complementarity assumption $v_{12}(\sigma,\tau) > 0$ and the inequalities $\sigma_{t+n+1} > \sigma_{t+n+2}$ and $\tau_t > \tau_{t+n+1}$ imply

$$\text{(A19)} \qquad v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}) > v(\sigma_{t+n+2}, \tau_t).$$

Substituting (A19) into (A18), we conclude that (A15) holds for $m=n+1$.

Proof of Proposition 2 Consider the vectors of runner-up buyers $(\sigma_t, \ldots, \sigma_{T+1})$ and $(\sigma'_t, \sigma'_{r(t)}, \ldots, \sigma'_{r^{\rho'_t}(t)})$. From Lemma 1 and the assumption $\tau'_t = \tau_t$, we get that $\sigma_t = \sigma'_t$. Moreover, from (1), we have that $\sigma_{T+1} = \sigma'_{r^{\rho'_t}(t)}$ and there exists an index $\ell(r^k(t)) \in \{t+1, \ldots, T+1\}$ such that $\sigma_{\ell(r^k(t))} = \sigma'_{r^k(t)}$ for $k = 0, \ldots, \rho'_t$, where $r^0(t) = t$. In other words, the characterization of the runner-up buyer (1) implies that the elements of the vector $(\sigma'_t, \sigma'_{r(t)}, \ldots, \sigma'_{r^{\rho'_t}(t)})$ are a subset of the elements of the vector $(\sigma_t, \sigma_{t+1}, \ldots, \sigma_{T+1})$. Lemma 1 then implies that $\tau_{\ell(r^k(t))} = \tau'_{r^k(t)}$ for $k = 0, \ldots, \rho'_t$. Therefore we can rewrite the payoff to seller τ'_t, as in (7), in the following way:

$$\text{(A20)} \qquad v(\sigma_{\ell(r(t))}, \tau_{\ell(t)}) - \sum_{k=1}^{\rho'_t} \Big[v(\sigma_{\ell(r^k(t))}, \tau_{\ell(r^k(t))}) - v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))}) \Big].$$

Now define δ_k to be an integer number such that $\ell(r^k(t)) + \delta_k = \ell(r^{k+1}(t))$. Then Lemma A1 implies that

$$\text{(A21)} \qquad \begin{aligned} & v(\sigma_{\ell(r^k(t))+1}, \tau_{\ell(r^k(t))}) - \sum_{h=1}^{\delta_k - 1} \Big[v(\sigma_{\ell(r^k(t))+h}, \tau_{\ell(r^k(t))+h}) - v(\sigma_{\ell(r^k(t))+h+1}, \tau_{\ell(r^k(t))+h}) \Big] \\ & > v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))}) \end{aligned}$$

for $k = 0, \ldots, \rho'_t - 1$. Substituting (A21) into (A20), we obtain (A14).

Proof of Proposition 3 We prove this result in two steps. We first show that if sellers choose investments $y(t,t+1)$, for $t = 1, \ldots, T$ (denoted *simple* investments), then the order of sellers' identities coincides with the order of sellers' qualities. Hence Corollary 1 applies, and the shares of the surplus accruing to each buyer and each seller are the ones defined in (8) and (9).

Step 1. If each seller t chooses the simple investment $y(t, t+1)$, as defined in (18), then

$$\tau_1 = \tau(1, y(1,2)) > \ldots > \tau_T = \tau(T, y(T, T+1)).$$

The proof follows from the fact that from the first-order conditions of (18), we obtain

$$\text{(A22)} \qquad \frac{\mathrm{d}\tau(t, y(t,s))}{\mathrm{d}t} = \frac{v_2 \tau_1 \tau_{22} - \tau_1 C'' - v_2 \tau_2 \tau_{12}}{v_{22}(\tau_2)^2 + v_2 \tau_{22} - C''} < 0$$

and

$$\text{(A23)} \qquad \frac{\mathrm{d}\tau(t, y(t,s))}{\mathrm{d}s} = \frac{v_{12}(\tau_2)^2}{v_{22}(\tau_2)^2 + v_2 \tau_{22} - C''} < 0,$$

where (with an abuse of notation) we denote by τ_h and τ_{hk}, $h,k \in \{1,2\}$, the first- and second-order derivatives of the quality functions $\tau(\cdot,\cdot)$ computed at $(t, y(t, s))$. Moreover, the first- and second-order derivatives (v_h and v_{hk}, $h, k \in \{1,2\}$) of the functions $v(\cdot,\cdot)$ are computed at $(\sigma_s,\tau(t,y(t,s)))$, and C'' is evaluated at $y(t,s)$.

We conclude the proof by showing that the sellers' choices of best replies $y(t,t + 1)$, $t + 1, \ldots, T$, are unique.

Step 2. The sellers' unique best replies in the investment game are $y(t,t + 1)$ for t=1, …, T. We start from seller T. In the Tth (the last) matching subgame of the Bertrand competition game, all sellers besides seller T have selected a buyer's bid. Denote by τ_T the quality of this seller. Assume for simplicity that $S = T + 1$. We use the same notation as in the proof of Proposition 1. In particular, since we want to show that seller T chooses a simple investment independently from the investment choice of the other sellers, we denote by $\alpha_{(T)}$ and $\alpha_{(T+1)}$ the qualities of the two buyers that are still unmatched in the Tth subgame, such that $\alpha_{(T)} > \alpha_{(T+1)}$. Indeed, from Lemma 1, the identities of the two buyers left will depend on the order of sellers' qualities and therefore on the investment choices of the other $(T-1)$ sellers.

From Lemma 1 we have that the buyer of quality $\alpha_{(T)}$ matches with seller T. The payoff of seller T is $v(\alpha_{(T+1)},\tau_T)$, while the payoff of the buyer of quality $\alpha_{(T)}$ is $[v(\alpha_{(T)}, \tau_T) - v(\alpha_{(T+1)}, \tau_T)]$, and the payoff of the buyer of quality $\alpha_{(T+1)}$ is zero.

Denote now by $a_{(T)}$, respectively $a_{(T+1)}$, the identity of the buyer of quality $\alpha_{(T)}$, respectively $\alpha_{(T+1)}$: $a_{(T)} < a_{(T+1)}$. The optimal investment y_T of seller T is then defined as

$$y_T = \operatorname{argmax}_y \; v(\alpha(T+1), \tau(T, y)) - C(y).$$

This implies that the optimal investment of seller T is the simple investment $y_T = y(T,a_{(T+1)})$, as defined in (18), whatever the pair of buyers left in the Tth subgame. If all other sellers undertake a simple investment, then from Step 1, $a_{(T)} = T$ and $a_{(T+1)} = T + 1$. Hence the optimal investment of seller T is $y(T,T + 1)$.

Denote now by $t + 1$ ($t < T$) the last seller who undertakes a simple investment $y(t + 1, t + 2)$. We then show that seller t will choose a simple investment $y(t,t + 1)$. Consider the tth subgame in which seller t has to choose among the potential bids of the remaining $(T - t + 2)$ buyers labelled $a_{(t)} < \ldots < a_{(T+1)}$, with associated qualities $\alpha_{(t)} > \ldots > \alpha_{(T+1)}$, respectively.[17] From the assumption that every seller $j = t + 1, \ldots, T$ undertakes a simple investment $y(j,a_{(j+1)})$, and Step 1, we obtain that $\tau_{t+1} > \ldots > \tau_T$.

We first show that the quality associated with seller t is such that $\tau_t > \tau_{t+1}$. Assume, by way of contradiction, that seller t chooses investment y^* that yields a quality τ^* such that $\tau_{j+1} \leq \tau^* \leq \tau_j$ for some $j \in \{t+1, \ldots, T - 1\}$. Then from Lemma 1 and (9) we have that seller t matches with buyer $a_{(j)}$, and the payoff of seller t is

$$\text{(A24)} \qquad \Pi^S_{\tau^*} = v(\alpha_{(j+1)}, \tau(t, y^*)) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right],$$

where $\tau(t, y^*) = \tau^*$. From (A24) we obtain that y^* is then the solution to the problem

$$\text{(A25)} \qquad y^* = \operatorname{argmax}_y \; v(\alpha(j+1), \tau(t, y)) - C(y).$$

From the assumption that each seller $j \in \{t+1, \ldots, T\}$ undertakes a simple investment, and definition (18), we also have that the investment choice $y(j,a_{(j+1)})$ of seller j is defined as

$$\text{(A26)} \qquad y(j, a_{(j+1)}) = \operatorname{argmax}_y \; v(\alpha(j+1), \tau(j, y)) - C(y).$$

Notice further that the payoff to seller t in (A24) is continuous in τ^*. Indeed, the limit for τ^* that converges from the right to τ_j is equal to

(A27) $$\Pi^S_{\tau_j} = v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right].$$

If instead $\tau_j < \tau^* \leq \tau_{j-1}$, then from (9) the payoff to the seller with quality τ^* is

(A28) $$\Pi^S_{\tau^*} = v(\alpha_{(j)}, \tau^*) - v(\alpha_{(j)}, \tau_j) + v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right].$$

Therefore the limit for τ^* that converges to τ_j from the left is, from (A28), equal to $\Pi^S_{\tau_j}$ in (A27). This proves the continuity in τ^* of the payoff function in (A24). Continuity of the payoff function in (A24), together with definitions (A25), (A26) and condition (A22), implies that $y^* > y(j, a_{(j+1)})$ or $\tau^* > \tau_j$, a contradiction to the hypothesis $\tau^* \leq \tau_j$.

We now show that seller t will choose a simple investment $y(t, a_{(t+1)})$. From the result just obtained we have $\tau_t > \tau_{t+1} > \ldots > \tau_T$, and the assumption that $\alpha_{(t)} > \ldots > \alpha_{(S)}$ are the qualities of the unmatched buyers in the tth subgame of the Bertrand competition game allows us to conclude, using (9), that the payoff to seller t is

$$\Pi^S_{\tau_t} = v(\alpha_{(t+1)}, \tau_t) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right].$$

The investment choice of seller t is then the simple investment $y(t, a_{(t+1)})$ defined as

(A29) $$y(t, a_{(t+1)}) = \text{argmax}_y \; v(\alpha(t+1), \tau(t, y)) - C(y).$$

In order to conclude that a simple investment $y(t, a_{(t+1)})$ is the unique solution to (A29), we still need to show that seller t has no incentive to deviate and choose an investment y*, and hence a quality τ^*, that exceeds the quality τ_k of one of the $(t-1)$ sellers that are already matched at the tth subgame of the Bertrand competition game: $k < t$. The reason why this choice of investment might be optimal for seller t is that it changes the pool of buyers $a_{(t)}, \ldots, a_{(S)}$ unmatched in subgame t. Of course, this choice will change the simple nature of the investment of seller t only if $\tau_k > \tau_{t+1}$. Indeed, we have already showed that if $\tau_k < \tau_{t+1}$, then $\tau_t > \tau_k$, and from (A29), the investment choice of seller t is $y_t(a_{(t+1)})$, a simple investment for any given set of unmatched buyers.

Consider the following deviation by seller t: seller t chooses an investment $y^* > y(t, a_{(t+1)})$ that yields quality $\tau^* > \tau_k > \tau_{t+1}$. Recall that Lemma 1 implies that the ranking of each seller in the ordered vector of sellers' qualities determines the buyer with whom each seller is matched. Hence the deviation of seller t changes the ranking and the matches of all sellers whose quality τ is smaller than τ^* and greater than τ_{t+1}. However, this deviation does not alter the ranking of the $T - t$ sellers with identities $(t+1, \ldots, T)$ and qualities $(\tau_{t+1}, \ldots, \tau_T)$. Therefore the only difference between the equilibrium set of unmatched buyers in the tth subgame and the set of unmatched buyers in the same subgame following the deviation of seller t is the identity and quality of the buyer who matches with seller t.[18] The remaining set of buyers' identities and qualities $(\alpha_{(t+1)}, \ldots, \alpha_{(S)})$ is unchanged. Hence following the deviation of seller t, the unmatched buyers' qualities are $\alpha^* > \alpha_{(t+1)} > \ldots > \alpha_{(T)}$, where α^* is the quality of the buyer that according to Lemma 1 is matched with seller t when the quality of this seller is τ^*. Equation (9) implies that the payoff of seller t following this deviation is then

(A30) $$\Pi^S_{\tau^*} = v(\alpha_{(t+1)}, \tau^*) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right].$$

Continuity of the payoff function in (A29) together with (A30) implies that the net payoff of seller t is maximized at $y(t,a_{(t+1)})$. Hence seller t cannot gain from choosing an investment $y^* > y(t, a_{(t+1)})$. This argument holds for $t < T$, implying that all sellers choose a simple investment. Therefore $a_{(t)}=t$ and the equilibrium investment choice of seller t is $y_t = y(t,t+1)$.

Proof of Corollary 2 The result follows from Proposition 3, the definition of efficient investment (18) when buyer t matches with seller t, and condition (A23).

Proof of Proposition 4 Notice first that if a central planner is constrained to choose the match between buyer s and seller t, then the constrained efficient investment of buyer s is the solution to the problem

$$\text{(A31)} \qquad x^*(s,t) = \text{argmax}_x \ v(\sigma(s,x),\tau_t) - C(x).$$

This investment $x^*(s,t)$ is defined by the following necessary and sufficient first-order conditions of (A31):

$$\text{(A32)} \qquad v_1(\sigma(s,x^*(s,t)),\tau_t)\ \sigma_2(s,x^*(s,t)) = C'(x^*(s,t)).$$

The result then follows from the observation that the definition of the constrained efficient investment $x^*(s,t)$, equation (A32), coincides with the definition of the optimal investment $x_s(t)$ of buyer s: equation (20).

Condition (20) implies that

$$\frac{\mathrm{d}\sigma(s,x_s(t))}{\mathrm{d}s} = \frac{\sigma_1 v_1 \sigma_{22} - \sigma_1 C'' - v_1\sigma_2\sigma_{12}}{v_{11}(\sigma_2)^2 + v_1\sigma_{22} - C''} < 0,$$
$$\frac{\mathrm{d}\sigma(s,x_s(t))}{\mathrm{d}t} = \frac{v_{12}(\sigma_2)^2}{v_{11}(\sigma_2)^2 + v_1\sigma_{22} - C''} < 0,$$

where the functions σ_h and σ_{hk}, $h,k \in \{1,2\}$, are computed at $(s,x_s(t))$; the functions v_h and v_{hk}, h, $k \in \{1,2\}$, are computed at $(\sigma(s,x_s(t)),\tau_t)$, and the second derivative of the cost function C'' is the second derivative of the cost function $C(\cdot)$ computed at $x_s(t)$.

Proof of Proposition 5 We prove this result in three steps. We first show that the buyers' equilibrium qualities $\sigma(i,x_i(i))$ associated with the equilibrium $s_i = i$ satisfy condition (21). We then show that the net payoff to buyer i associated with any given quality σ of this buyer is continuous in σ. This result is not obvious since from Lemma 1—given the investment choices of other buyers—buyer i can change his equilibrium match by changing his quality σ. Finally, we show that this net payoff has a unique global maximum, and this maximum is such that the corresponding quality σ is in the interval in which buyer i is matched with seller i. These steps clearly imply that each buyer i has no incentive to deviate and choose an investment different from the one that maximizes his net payoff and yields an equilibrium match with seller i.

Let $\pi_i^B(\sigma) - C(x(i,\sigma))$ be the net payoff to buyer i, where $x(i,\sigma)$ denotes the investment level of buyer i associated with quality σ:

$$\text{(A33)} \qquad \sigma(i,x(i,\sigma)) \equiv \sigma.$$

Step 1. The equilibrium quality $\sigma(i,x_i(i))$ of buyer i is such that $\sigma(i,x_i(i)) = \sigma_i < \sigma(i-1,x_{i-1}(i-1)) = \sigma_{i-1}$ for all $i = 2, \ldots, S$.

The proof follows directly from Proposition 4.

Step 2. The net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Let $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_S)$ be the given ordered vector of the qualities of the buyers, other than i. Notice that if $\sigma \in (\sigma_{i-1}, \sigma_{i+1})$, then by Lemma 1 buyer i is matched with the seller of quality τ_i. Then by Corollary 1 and the definition of $v(\cdot,\cdot)$, $C(\cdot)$, $\sigma(\cdot,\cdot)$ and (A33), the payoff function $\pi_i^B(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Consider now the limit for $\sigma \to \sigma_{i-1}^-$ from the right of the net payoff to buyer i when it is matched with the seller of quality τ_i, $\sigma \in (\sigma_{i+1}, \sigma_{i-1})$. From (8), this limit is

$$\begin{aligned} &\pi_i^B(\sigma_{i-1}^-) - C(x(i,\sigma_{i-1}^-)) \\ &= v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) + \sum_{h=i+1}^{T}[v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma_{i-1})). \end{aligned} \tag{A34}$$

Conversely, if $\sigma \in (\sigma_{i-1}, \sigma_{i-2})$, then by Lemma 1, buyer i is matched with the seller of quality τ_{i-1} and the payoff is continuous in this interval. Then from (8), the limit for $\sigma \to \sigma_{i-1}^+$ from the left of the net payoff to buyer i when matched with the seller of quality τ_{i-1} is

$$\begin{aligned} \pi_i^B(\sigma_{i-1}^+) - C(x(i,\sigma_{i-1}^+)) &= v(\sigma_{i-1},\tau_{i-1}) - v(\sigma_{i-1},\tau_{i-1}) + v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) \\ &\quad + \sum_{h=i+1}^{T}[v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma_{i-1})). \end{aligned} \tag{A35}$$

In the latter case, while the buyer of quality σ is matched with the seller of quality τ_{i-1}, the buyer of quality σ_{i-1} is matched with the seller of quality τ_i. Equation (A34) coincides with equation (A35) since the first two terms on the left-hand side of equation (A35) are identical. A similar argument shows continuity of the net payoff function at $\sigma = \sigma_h$, $h = 1, \ldots, i-2, i+1, \ldots, N$.

Step 3. The net surplus function $\pi_i^B(\sigma) - C(x(i,\sigma))$ has a unique global maximum in the interval $(\sigma_{i+1}, \sigma_{i-1})$.

Notice that in the interval $(\sigma_{i+1}, \sigma_{i-1})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is given by

$$\pi_i^B(\sigma) - C(x(i,\sigma)) = v(\sigma,\tau_i) - v(\sigma_{i+1},\tau_i) + \sum_{h=i+1}^{T}[v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma)). \tag{A36}$$

This expression, and therefore the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$, is strictly concave in σ (by strict concavity of $v(\cdot,\tau_i)$ and $\sigma(i,\cdot)$, and strict convexity of $C(\cdot)$) in the interval $(\sigma_{i+1}, \sigma_{i-1})$, and reaches a maximum at $\sigma_i = \sigma(i, x_i(i))$ as defined in (20). Notice further that in the right-adjoining interval $(\sigma_{i-1}, \sigma_{i-2})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is given by the following expression (different from (A36)):

$$\begin{aligned} \pi_i^B(\sigma) - C(x(i,\sigma)) &= v(\sigma,\tau_{i-1}) - v(\sigma_{i-1},\tau_{i-1}) + v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) \\ &\quad + \sum_{h=i+1}^{T}[v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma)). \end{aligned} \tag{A37}$$

This new expression of the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is also strictly concave (by strict concavity of $v(\cdot,\tau_{i-1})$ and $\sigma(i,\cdot)$, and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i, x_i(i-1))$. From Proposition 4 we know that

$$\sigma(i, x_i(i-1)) < \sigma_{i-1} = \sigma(i-1, x_{i-1}(i-1)).$$

This implies that in the interval $(\sigma_{i-1}, \sigma_{i-2})$, the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ.

A symmetric argument shows that the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ in any interval (σ_h, σ_{h-1}) for h=2, …, i−2.

Notice further that in the left-adjoining interval $(\sigma_{i+2}, \sigma_{i+1})$, by Lemma 1 and Proposition 1, the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ takes the following expression (different from (A36) and (A37)):

$$\pi_i^B(\sigma) - C(x(i,\sigma))$$
$$= v(\sigma, \tau_{i+1}) - v(\sigma_{i+2}, \tau_{i+1}) + \sum_{h=i+2}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] - C(x(i,\sigma)).$$

This new expression of the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is also strictly concave in σ (by strict concavity of $v(\cdot,\tau_{i+1})$ and $\sigma(i,\cdot)$, and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i, x_i(i+1))$, which from Proposition 4 is such that $\sigma_{i+1} = \sigma(i+1, x_{i+1}(i+1)) < \sigma(i, x_i(i+1))$. This implies that in the interval $(\sigma_{i+2}, \sigma_{i+1})$, the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ.

A symmetric argument shows that the net payoff $\pi_i^B(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ in any interval (σ_{k+1}, σ_k) for $k = i+2, \ldots, T-1$.

Proof of Proposition 6 First, for a given ordered vector of sellers' quality functions ($\tau(1,\cdot), \ldots, \tau(T,\cdot)$), we construct an inefficient equilibrium of the buyers' investment game such that there exists one buyer, labelled s_j, $j \in \{2, \ldots, S\}$, such that $s_j < s_{j-1}$.

In order to show that a vector $(s_1, \ldots, s_j, \ldots, s_S)$ is an equilibrium of the buyers' investment game, we need to verify that condition (21) holds for $i = 2, \ldots, S$, and no buyer s_i has an incentive to deviate and choose an investment x different from $x_{s_i}(i)$, as defined in (19).

Notice first that for every buyer other than s_j and s_{j-1}, Proposition 5 applies, hence it is an equilibrium for each buyer to choose investment level $x_{s_i}(i)$, as defined in (19), such that (21) is satisfied. We can therefore restrict attention to buyers s_j and s_{j-1}. In particular, we need to consider a buyer s_{j-1} of a quality arbitrarily close to that of buyer s_j. This is achieved by considering a sequence of quality functions $\sigma^n(s_{j-1}, \cdot)$ that converges uniformly to $\sigma(s_j, \cdot)$.[19] Then from definition (19), the continuity and strict concavity of $v(\cdot,\tau)$ and $\sigma(s,\cdot)$, the continuity and strict convexity of $C(\cdot)$, and the continuity of $v_1(\cdot,\tau)$, $\sigma_2(s,\cdot)$ and $C'(\cdot)$, for any given $\varepsilon > 0$, there exists an index n_ε such that for $n > n_\varepsilon$,

$$|\sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)) - \sigma(s_j, x_{s_j}(j-1))| < \varepsilon. \tag{A38}$$

From Proposition 4 and the assumptions $s_j < s_{j-1}$, we also know that for $n > n_\varepsilon$,

$$\sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)) < \sigma(s_j, x_{s_j}(j-1)), \tag{A39}$$

while from the assumption $\tau_j < \tau_{j-1}$, we have

$$\sigma(s_j, x_{s_j}(j)) < \sigma(s_j, x_{s_j}(j-1)). \tag{A40}$$

Inequalities (A38), (A39) and (A40) imply that for any buyer s_{j-1} characterized by the quality function $\sigma^n(s_{j-1}, \cdot)$, where $n > n_\varepsilon$, the equilibrium condition (21) is satisfied:

$$\text{(A41)} \qquad \sigma(s_j, x_{s_j}(j)) < \sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)).$$

In order to conclude that $(s_1, \ldots, s_j, \ldots, s_S)$ is an equilibrium of the buyers' investment game, we still need to show that neither buyer s_j nor buyer s_{j-1} wants to deviate and choose an investment different from $x_{s_j}(j)$ and $x_{s_{j-1}}(j-1)$, where the quality function associated with buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ for $n > n_\varepsilon$. Consider the net payoff to buyer s_j, namely $\pi^B_{s_j}(\sigma) - C(x(s_j, \sigma))$. An argument symmetric to the one used in Step 2 of the proof of Proposition 5 shows that this payoff function is continuous in σ. Moreover, from the notation of σ_j in Section III, Proposition 4, (A39) and (A41), we obtain that $\sigma_j < \sigma^n_{j-1} < \sigma(s_j, x_{s_j}(j-1)) < \sigma_{j-2}$. Then using an argument symmetric to the one used in Step 3 of the proof of Proposition 5, we conclude that this net payoff function has two local maxima, at σ_j and $\sigma(s_j, x_{s_j}(j-1))$, and a kink at σ^n_{j-1}. We then need to show that there exists at least one element of the sequence σ^n_{j-1} such that the net payoff $\pi^B_{s_j}(\sigma) - C(x(s_j, \sigma))$ reaches a global maximum at σ_j. Then when the quality function of buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$, buyer s_j has no incentive to deviate and choose a different investment.

From (8), the net payoff $\pi^B_{s_j}(\sigma) - C(x(s_j, \sigma))$ computed at σ_j is greater than the same net payoff computed at $\sigma(s_j, x_{s_j}(j-1))$ if and only if

$$\text{(A42)} \qquad \begin{aligned} v(\sigma_j, \tau_j) - C\big(x(s_j, \sigma_{(j)})\big) \geq{}& v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) - v(\sigma^n_{j-1}, \tau_{j-1}) \\ &+ v(\sigma^n_{j-1}, \tau_j) - C\Big(x\Big(s_j, \sigma(s_j, x_{s_j}(j-1))\Big)\Big). \end{aligned}$$

Inequality (A38) and the continuity of $v(\cdot, \tau_{j-1})$, $\sigma(s_j, \cdot)$ and $C(\cdot)$ imply that for any given $\varepsilon > 0$, there exists a ξ_ε and an n_{ξ_ε} such that for $n > n_{\xi_\varepsilon}$ we have

$$|v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) - v(\sigma^n_{j-1}, \tau_{j-1})| < \xi_\varepsilon$$

and

$$|C\Big(x\Big(s_j, \sigma(s_j, x_{s_j}(j-1))\Big)\Big) - C\big(x(s_j, \sigma^n_{j-1})\big)| < \xi_\varepsilon.$$

These two inequalities imply that a necessary condition for (A42) to be satisfied is

$$\text{(A43)} \qquad v(\sigma_j, \tau_j) - C(x(s_j, \sigma_j)) \geq v(\sigma^n_{j-1}, \tau_j) - C(x(s_j, \sigma^n_{j-1})) + 2\xi_\varepsilon.$$

We can now conclude that there exists an $\varepsilon > 0$ such that for $n > n_{\xi_\varepsilon}$, condition (A43) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot, \tau_j)$ and $\sigma(s_j, \cdot)$, and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_j) - C(x(s_j, \sigma))$ is strictly concave and has a unique interior maximum at σ_j.

Consider now the net payoff to buyer s_{j-1}, namely $\pi^B_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$. An argument symmetric to the one used above allows us to prove that this payoff function is continuous in σ. Further, from the notation of σ_j in Section III, Proposition 4 and (A41), we have that $\sigma_{j+1} < \sigma^n(s_{j-1}, x_{s_{j-1}}(j)) < \sigma_j < \sigma^n_{j-1}$. Therefore we conclude that the net surplus function $\pi^B_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ has two local maxima, at σ^n_{j-1} and $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$, and a kink at σ_j. We still need to prove that there exists at least one element of the sequence σ^n_{j-1} such that the net payoff $\pi^B_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ reaches a global maximum at σ^n_{j-1}, which implies that when the quality function of buyer s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$, this buyer has no incentive to deviate and choose a different investment.

From (8), the net payoff $\pi^B_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ computed at σ^n_{j-1} is greater than the same net payoff computed at $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ if and only if

$$\text{(A44)} \qquad \begin{aligned} &v(\sigma^n_{j-1}, \tau_{j-1}) - v(\sigma_j, \tau_{j-1}) + v(\sigma_j, \tau_j) - C(x(s_{j-1}, \sigma^n_{j-1})) \\ &\geq v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j) - C\Big(x\Big(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j))\Big)\Big). \end{aligned}$$

Definition (19), the continuity and strict concavity of $v(\cdot,\tau_j)$ and $\sigma(s_{j-1},\cdot)$, the continuity and strict convexity of $C(\cdot)$, and the continuity of $v_1(\cdot,\tau_j)$, $\sigma_2(s_j,\cdot)$ and $C'(\cdot)$, imply that for given $\varepsilon' > 0$ there exists an $n_{\varepsilon'}$, a $\xi_{\varepsilon'}$ and an $n_{\xi_{\varepsilon'}}$ such that for $n > n_{\varepsilon'}$ we have

$$|\sigma^n(s_{j-1}, x_{s_{j-1}}(j)) - \sigma_j| < \varepsilon',$$

while for $n > n_{\xi_{\varepsilon'}}$ we have

$$|v(\sigma_j, \tau_j) - v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j)| < \xi_{\varepsilon'}$$

and

$$|C(x(s_{j-1}, \sigma_j)) - C\Big(x\Big(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j))\Big)\Big)| < \xi_{\varepsilon}.$$

The last two inequalities imply that a necessary condition for (A44) to be satisfied is

$$\text{(A45)} \qquad v(\sigma^n_{j-1}, \tau_{j-1}) - C(x(s_{j-1}, \sigma^n_{j-1})) \geq v(\sigma_j, \tau_{j-1}) - C(x(s_{j-1}, \sigma_j)) + 2\xi_{\varepsilon'}.$$

We can now conclude that there exists an $\varepsilon' > 0$ such that for $n > n_{\xi_{\varepsilon'}}$, condition (A45) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot,\tau_{j-1})$ and $\sigma^n(s_{j-1},\cdot)$, and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_{j-1}) - C(x(s_{j-1}, \sigma))$ is strictly concave and has a unique interior maximum at σ^n_{j-1}. This concludes the construction of the inefficient equilibrium of the buyers' investment game.

We need now to show that for any given vector of buyers' quality functions ($\sigma(s_1,\cdot)$, …, $\sigma(s_S,\cdot)$), it is possible to construct an ordered vector of sellers' quality functions ($\tau(1,\cdot)$, …, $\tau(T,\cdot)$) such that no inefficient equilibrium exists.

Assume, by way of contradiction, that an inefficient equilibrium exists for any ordered vector of sellers' quality functions ($\tau(1,\cdot)$, …, $\tau(T,\cdot)$). Consider first the case in which this inefficient equilibrium is such that there exists only one buyer s_j such that $s_j < s_{j-1}$. Let $\tau^n(j-1,\cdot)$ be a sequence of quality functions for seller $(j-1)$ such that $\tau^n(j-1,y) > \tau(j,y)$ for all y, and $\tau^n(j-1,\cdot)$ converges uniformly to $\tau(j,\cdot)$. From Proposition 4 and the assumption $s_j < s_{j-1}$, we have

$$\text{(A46)} \qquad \sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)),$$

where $x_{s_j}(j)$ and $x_{s_{j-1}}(j)$ are defined in (19). Further, denote by $x^n_{s_{j-1}}(j-1)$ the optimal investment defined, as in (20), by the following set of first-order conditions:

$$v_1(\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1})\sigma_2(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1}) = C'(x^n_{s_{j-1}}(j-1)).$$

Then from Proposition 4 we have

$$\text{(A47)} \qquad \sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)).$$

Further, continuity of the functions $v(\sigma,\cdot)$, $v_1(\sigma,\cdot)$, $\sigma(s,\cdot)$, $\sigma_2(s,\cdot)$, $C(\cdot)$ and $C'(\cdot)$ implies that for given $\widehat{\varepsilon} > 0$, there exists an $n_{\widehat{\varepsilon}}$ such that for $n > n_{\hat{\varepsilon}}$,

$$\text{(A48)} \qquad |\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) - \sigma(s_{j-1}, x_{s_{j-1}}(j))| < \widehat{\varepsilon}.$$

Then from (A46), (A47) and (A48) there exists an $\widehat{\varepsilon} > 0$ and hence an $n_{\widehat{\varepsilon}}$ such that for $n > n_{\hat{\varepsilon}}$,

$$\text{(A49)} \qquad \sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)).$$

Inequality (A49) clearly contradicts the necessary condition (21) for the existence of the inefficient equilibrium.

A similar construction leads to a contradiction in the case where the inefficient equilibrium is characterized by more than one buyer s_j such that $s_j < s_{j-1}$.

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NOTES

1. Notice that Ramey and Watson (2001) also consider how matching frictions can alleviate the inefficiencies due to the hold-up problem in the presence of incomplete contracts, and match specific investments in an ongoing repeated relationship. See also Ramey and Watson (1997) for a related result.
2. For simplicity, we take both cost functions to be identical; none of our results depends on this assumption. If the cost functions were type-specific, then we would require the marginal costs to increase with the identity of the buyer or the seller.
3. For convenience, both $\sigma(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ are assumed to be twice differentiable on $[1, S] \times \mathbb{R}_+$.
4. For convenience, we write $v_l(\cdot,\cdot)$ for the partial derivative of the surplus function $v(\cdot,\cdot)$ with respect to the lth argument and write $v_{lk}(\cdot,\cdot)$ for the cross-partial derivative with respect to the lth and kth arguments, or the second-partial derivatives if $l = k$. We use the same notation for the functions $\sigma(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ defined above.
5. As established in Milgrom and Roberts (1990, 1994) and Edlin and Shannon (1998), our results can be derived with much weaker assumptions on the smoothness and concavity of the surplus function $v(\cdot,\cdot)$ and the two quality functions $\sigma(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ in the two investments x_s and y_t.
6. See Felli and Roberts (2001) for a discussion of the case in which sellers select their bids in the order of any permutation of the sellers' identities $(1, \ldots, T)$.
7. The dynamic version of the same equilibrium notion has been used in the analysis of Bergemann and Välimäki (1996) and Felli and Harris (1996).
8. This modification of the extensive form is equivalent to a Bertrand competition model in which there exists an indivisible smallest possible unit of a bid (a penny) so that each buyer can break any tie by bidding one penny more than his opponent if he wishes to do so.
9. Notice that given the notation defined here, it is not necessarily the case that $\sigma_t > \sigma_{t+1} > \ldots > \sigma_T$.
10. Notice that all unmatched buyers with a strictly positive willingness to pay for the match with a given seller submit their bids in equilibrium.
11. Recall that since $\tau_1 > \ldots > \tau_T$, Lemma 1 and the notation defined in Section III imply that $\sigma_1 > \ldots > \sigma_S$.
12. The level of investment $x(i,\sigma)$ is defined as in the Appendix: $\sigma(i,x) \equiv \sigma$.
13. Indeed, from (A34) and (A35) in the Appendix, we get that

$$\frac{\partial\left[\pi_i^B(\sigma_{i-1}^-) - C\left(x(i, \sigma_{i-1}^-)\right)\right]}{\partial\sigma} = v_1(\sigma_{i-1}, \tau_i) - \frac{C'(x(i, \sigma_{i-1}))}{\sigma_2(i, x(i, \sigma_{i-1}))}$$

and

$$\frac{\partial\left[\pi_i^B(\sigma_{i-1}^+) - C\left(x(i,\sigma_{i-1}^+)\right)\right]}{\partial\sigma} = v_1(\sigma_{i-1},\tau_{i-1}) - \frac{C'(x(i,\sigma_{i-1}))}{\sigma_2(i,x(i,\sigma_{i-1}))}.$$

Therefore from $v_{12}(\sigma,\tau) > 0$, we conclude that

$$\frac{\partial\left[\pi_i^B(\sigma_{i-1}^+) - C\left(x(i,\sigma_{i-1}^+)\right)\right]}{\partial\sigma} > \frac{\partial\left[\pi_i^B(\sigma_{i-1}^-) - C\left(x(i,\sigma_{i-1}^-)\right)\right]}{\partial\sigma}.$$

14. See Felli and Roberts (2001) for the formal statement and proof of this result.
15. Notice that we can conclude that following a deviation by buyer σ'', the bid accepted by seller τ_j is not higher than $b(\tau_j)$ since, as discussed in Section II, we allow buyers to specify in their bid that they are willing to increase such a bid if necessary. Moreover, we restrict the strategy used by each seller so as to put higher-order probabilities on the bids that contain this proviso. In the absence of these restrictions it is possible to envisage a situation in which following a deviation by buyer σ'', the sellers who select their bid after seller τ_i and before seller τ_j may no longer choose among equal bids the one submitted by the buyer with the highest willingness to pay. The result is then that the bid accepted by seller τ_j following a deviation might actually be higher than $b(\tau_j)$. Notice that this problem disappears if we assume that there exists a smallest indivisible unit of a bid (see also note 8).
16. This is just one of a whole continuum of subgame-perfect equilibria of this simple Bertrand game, *but* it is the unique cautious equilibrium.
17. Once again we want to show that seller t undertakes a simple investment independently of the investment choice of sellers 1, …, $t-1$ that, from Lemma 1, determines the exact identities of the unmatched buyers in the tth subgame of the Bertrand competition game.
18. All other sellers with identities (k, …, $t-1$) whose match changed because of the deviation are already matched in the tth subgame of the Bertrand competition game.
19. The sequence $\sigma^n(s_{j-1},\cdot)$ converges uniformly to $\sigma(s_j,\cdot)$ if and only if

$$\lim_{n\to\infty}\sup_x|\sigma^n(s_{j-1},x) - \sigma(s_j,x)| = 0.$$

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