INCOMPLETE WRITTEN CONTRACTS: UNDESCRIBABLE STATES OF NATURE*

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This paper explores the extent to which the incompleteness of contracts can be attributed to their *formal nature:* the form, usually written, that contracts are required to take to be enforceable in a court of law by legal prescription, common practice, or simply the contracting parties' will. We model the formal nature of state-contingent contracts as the requirement that the mapping from states of the world to the corresponding outcomes must be of an *algorithmic* nature. It is shown that such algorithmic nature, although by itself is not enough to generate incomplete contracts, when paired with a similar restriction on the contracting parties' selection process yields *endogenously incomplete* optimal contracts.

I. INTRODUCTION

I.1. Overview

Many contracts specifying state-contingent outcomes that economic agents write to regulate their transactions are *incomplete.* They neglect information about the state of the world that is in principle available to the parties to the contract and would be optimal for them to include. This paper explores the extent to which contracts' incompleteness could be attributed to the *formal nature* of contracts: the form, usually written, that contracts are required to take by legal prescription, common practice, or simply the contracting parties' will, to be enforceable in a court of law.[1]

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1. For example, in American common law a number of contracts require the *formality of writing* to be enforceable. These contracts are listed in the Statute of Frauds and the Sales Act, modified by the Uniform Commercial Code, and include:

> ". . . any special promise, to answer damages out of his own estate; . . . any special promise to answer for the debt, default, or miscarriage of another person; . . . any agreement made upon consideration of marriage; . . . any contract or sale of lands, tenements or hereditaments, or any interest in or concerning them; . . . any agreement that is not to be performed within the space of one year from the making thereof; . . . [any] contract for the sale of any goods, wares and merchandize, for the price of [$500 or more] . . .

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Intuitively, it is clear that not every agreement can be written in a contract. For example, no contract can be infinitely long. Hence, every agreement that requires an infinitely long description, say of the circumstances in which the agreement applies, cannot be written, and consequently enforced. Consider a written contract consisting of a finite number of "clauses." Once the state of nature is realized, the parties (or the enforcing agency) will examine the available "evidence" about the state of the world that has occurred and identify which clause(s) apply to the case at hand and hence what outcome is prescribed by the contract. We take the view that this means that the mapping between states and outcomes that the contract embodies must be of an *algorithmic* nature. In this paper we examine the consequences of formal imposing this restriction on state-contingent contracts.

The formal notion of *algorithmic* which we adopt is that of *general recursive* functions or *effectively computable* functions. Intuitively, a function is effectively computable if there exists a *finite* device (an algorithm) that is capable of computing each of its values in a *finite* number of steps. There is a general consensus in the mathematical literature that the class of general recursive functions captures the widest possible intuitive notion of effective computability.[2] The class of general recursive functions coincides with the class of functions that can be computed by a class of abstract computing devices known as *Turing machines*. In the paper we model the *formal nature* of contracts requiring the mapping between states of the world and prescribed outcomes to be general recursive, or equivalently to be computable by a Turing machine.[3]

Once the restriction of computability is introduced, it is not difficult to see that some contracts simply cannot be written. This is equivalent to the statement that not all functions are in fact computable. Consider, for instance, a contract that prescribes two distinct outcomes according to whether the state of nature takes

> except the buyer shall accept part of the goods so sold, and actually receive the same, or give something in earnest to bind the bargain, or in part of payment, . . ."

The agreement for these transactions ". . . shall be in writing, and signed by the party to be charged therewith, or some other person thereunto by him lawfully authorized" [Calamari and Perillo 1987, p. 775].

2. See, for instance, Davis [1958], Rogers [1967], and Cutland [1980], or for a brief exposition, Anderlini [1989].

3. Throughout the paper, we shall use the words general recursive, effectively computable, and computable (by a Turing machine) in an equivalent way.

one particular critical value (or set of values) or not.[4] In this case a computable description of the contract necessarily contains an exact description of the critical state. Therefore, the fact that there exist states of nature (or events) which cannot be fully described by a finite algorithm directly entails that some contracts are not computable, and hence cannot be written.

The impossibility to write down some contracts, however, is not enough to generate genuinely endogenously incomplete contracts. The reason is that, under mild assumptions, it turns out that any contract (in a very general class) can always be approximated—in terms of the expected utility that it yields to the parties—by a sequence of computable ones. This is what we call the *approximation result.* An intuitive account is as follows.

In state-contingent contracting problems, some very natural *continuity* and boundedness properties of the contracting parties' preferences hold. Given a state of nature, in fact, it is hard to argue that the utility of either party is anything but continuous in the "share of the pie" which the contract prescribes. In addition, the contracting parties are interested only in the *expected* utility they derive from the contract, and a computable contract can always be constructed to partition the state space into an arbitrarily fine "grid," and to yield any desired *approximate* share of the pie to either party. Hence, in expected utility terms it is always possible to approximate the first best with a computable contract, even if the first-best contract itself is *not* computable. Thus, on the one hand, parties are not satisfied with any contract they can write; on the other hand, the only contract that satisfies the parties cannot be written since it is not computable. In formal terms, the expected utility maximization problem of choosing a computable contract does not have a solution. We expand considerably on the approximation result in Section IV below.

The approximation result is one way to read the contribution of this paper. In the absence of specific "complexity costs" such as the cost of writing "longer contracts" or perhaps the costs of making "more complex" statements within the text of a contract, the restriction that the parties must be capable of writing down their agreement is *not* enough to generate genuinely incomplete contracts. As we mentioned above, this hinges crucially on some natural continuity and boundedness properties of a state-

4. For instance, in a standard insurance or coinsurance contract, the critical state may be the "accident" state.

contingent contracting problem. When the object of the contract is the "performance" of either, or both, parties, continuity may become a less compelling assumption. Hence, the approximation result may be destroyed, and the computability restriction by itself may generate genuinely incomplete contracts. We explore the nature of written contracts over parties' performance in a companion paper [Anderlini and Felli 1993b].

The main result of this paper, however, is that under some additional restrictions which we view as plausible, agents will indeed write incomplete state-contingent contracts. To generate this result, we couple the restriction that contracts must be of an algorithmic nature with a similar restriction on the agents' contract selection process. The formality of the legal system requires parties, or their lawyers, when choosing among different contracts to be ready to present *formal* arguments to support their choice. We model this feature requiring the process by which contracts are chosen to be itself algorithmic or computable. With this additional restriction we are able to show (see Section V) that there exist cases in which the parties will end up writing an *endogenously incomplete* contract. The intuition behind this second result is as follows.

Consider a contracting problem in which the first best identifies one critical state of nature. Consider now a sequence of computable contracts that approximates the parties' first-best expected utility. From this sequence of computable contracts we can recover "finer and finer" approximations of the critical state of nature. If a computability restriction is imposed on the selection process, the resulting approximating sequence of computable contracts is itself computable. Hence, it follows that we must be able to approximate in a *computable* way the critical state. Since there exist states of nature (or events) which *cannot* be approximated in a computable way, it follows that it is not always possible to approximate the parties' first-best expected utility. The result is a genuinely incomplete contract.

We call the states of nature that cannot be approximated in a computable way *algorithmically undescribable* or simply undescribable states. Intuitively, a state (or an event) is undescribable, if it is not possible to generate with a finite algorithm, a list of its "characteristics." Any attempt to use a finite algorithm to describe an undescribable state, will inevitably leave out some relevant parts of the exact description of the state. We find this to be consistent with the intuitive notion of an incomplete contract.

We define an incomplete contract as one that takes into account less than the available information to the parties which—in a world where no restrictions (algorithmic or otherwise) are imposed on contracts—it would be optimal for the parties to include. We make this notion precise and discuss it further in Section III below. Contracts are incomplete because the parties lack the technology to describe in sufficient detail some critical states of nature and hence leave some valuable information out of the contract. This is intuitively consistent with the cause of contract incompleteness mentioned, for instance, in Hart and Moore [1988].

In virtually all the available literature the operational definition of an incomplete contract is that of a contract which is *silent*, makes no prescriptions, for some states of nature.[5] On the face of it, this seems quite radically different from the formal definition of incompleteness we put forward here. We only consider contracts that do make a prescription for all possible states of nature. Further, we define incompleteness as a property of the *partition* of the possible states of nature that the contract induces since it is this partition which characterizes how much of the relevant information is included in the contract itself. However, we believe that previous definitions and the one we propose are not in contrast. Insofar as the parties entering a silent contract are aware of the ex post mechanism that will operate when uncontracted contingencies arise, it is always possible to view a silent contract, together with the associated ex post mechanism, as a contract which makes prescriptions for all states of nature, but which is *constrained* to use the ex post mechanism for some relevant subset of the possible states. If it is impossible to perfectly "fine tune" the ex post mechanism, this contract may end up not distinguishing sufficiently finely among some relevant states of nature. It may end up including less information about the states of nature than would be optimal for the parties to include. Hence, it may end up being incomplete according to the definition proposed in this paper. We return to the relationship between the definition of incompleteness proposed here and silent contracts after we make our definition explicit in Section III below.

Courts play a completely passive role in the world we envisage here. We focus on situations in which contracts must be written to be enforceable. The enforcement mechanism is completely mechani-

5. Allen and Gale [1992] and Spier [1992] are notable exceptions.

cal, however. The courts (if this is the enforcement mechanism in use) simply ascertain what the contract prescribes after a state of nature is realized and ensure that the prescription is enforced. Given that the contracts we obtain are incomplete, in the sense that they include less information than would be optimal in a first-best world, this leaves open the possibility that our approach may be used to explain an active role for the courts which has proved elusive to previous work. In fact, if the contracting parties are constrained by what they can *write* into a contract, but the court can make "finer" distinctions than they can, it is possible that they will intentionally rely on the court's "interpretation" of their agreement. A model of the court's behavior, and of the contracting parties' ability to manipulate and forecast it, would need to be grafted onto the analysis we carry out here.

The material is organized as follows. The basic model is presented in Section II. Section III formally introduces our definition of contract incompleteness. Section IV concentrates on the approximation result mentioned above. The main result of the paper on endogenously incomplete contracts is presented in Section V. In this same section we also characterize the form that optimal incomplete contracts take in the case of a simple coinsurance problem. Section VI offers some concluding remarks. An Appendix contains some of the proofs. Before moving to the basic model we briefly discuss some related literature.

I.2. Related Literature

Since the Simon [1951] and, more recently, the Grossman and Hart [1986] seminal contributions, a number of papers have discussed incomplete contracts. The main branch of this literature *assumes* that contracts are incomplete and proceeds to analyze the *consequences* of incompleteness on the economy. This literature concentrates on the role of available mechanisms and institutions in mitigating the inefficiencies generated by contract incompleteness, such as vertical and lateral integration [Grossman and Hart 1986] and the optimal allocation of ownership rights on physical capital [Hart and Moore 1990]. Our model differs from these papers since we do not assume contract incompleteness but we derive it *endogenously* from the restriction imposed on the set of possible agreements by the algorithmic nature of a contract and of the selection process.

Hart and Moore [1988] and a number of subsequent papers [Chung 1991; Aghion, Dewatripont, and Rey 1994; Nöldeke and

Schmidt 1994] ask the question of whether one of the causes of contract incompleteness is the fact that the outcome that the parties wish to implement through a contract may be, at least in part, unobservable by the enforcing agency (the court). They conclude that the parties will write a silent contract; it will leave out some details that the court cannot observe. Whether the implemented outcome will differ from the socially efficient one, seems to be highly dependent on how much exactly the court observes of the contracted transaction. In our framework, all the parties (enforcing agency included) can, in principle, observe all the relevant variables. They are constrained only by what they can *write* into the contract.

Finally, few recent papers have derived endogenously contract incompleteness focusing on complementary, but different from ours, causes for incompleteness: the costs of specifying contingencies [Dye 1985]; the signaling effect—in a world of asymmetric information—of the parties' willingness to include a contingency in the contract [Hermalin 1988; Aghion and Hermalin 1990; Allen and Gale 1992; Spier 1992]; limited rationality defined in terms of psychological costs of evaluating the consequences of the parties' actions and decisions [Lipman 1993], and finally the contracting parties' strategic advantage from the specification of an incomplete contract in the first of a multistage contract bargaining procedure [Busch and Horstmann 1992].

II. The Model

II.1. The General Problem

We consider a very simple contracting problem. Two risk-averse parties agree ex ante on a contract that allows them to share the risk associated with the common random environment in which they operate. Let the two parties be indexed by $i = 1, 2$ and endowed with utility functions of the *consequence* c, assumed for simplicity to be one-dimensional $V_i(c)$. The state of nature s indexes the random environment parties face and takes values on a continuous, convex, and compact state space, "normalized" to be the closed unit interval $\mathscr{S} = [0,1]$.[6] The parties' problem is to find an agreement that specifies a sharing rule $x(\cdot)$ of the common surplus yielding $f_i(x(s);s)$ as the consequence to party i in state of nature s. To simplify the statement of the problem, we can define

6. All the results presented in our analysis generalize to the case of a countably infinite state space. We refer the interested reader to Anderlini and Felli [1993a].

the following indirect utility function of sharing rule $x(s)$ and state of nature s: $U_i[x(s);s] \equiv V_i[f_i(x(s);s)]$. For example, in the case in which the state of nature affects the utility of each party changing the size of the surplus $\pi(s)$ which the parties are agreeing to split, the indirect utilities take the form $U_i[x(s)]$ for one party and $U_i[\pi(s) - x(s)]$ for the other.

We assume $U_i(x;s)$ to be bounded from above and from below and continuous in x for any given $s \in \mathcal{S}$. We also assume that the two parties have ex ante symmetric but incomplete information concerning the state of nature s: they share the common prior probability measure $\mu(\cdot)$ with support $\mathcal{S}$. Throughout the rest of the paper we focus on properties that are valid up to a μ-measure zero set of states of nature. So two sharing rules are considered equal if they are the same except (possibly) for a set of states that has μ-measure zero.

We use a fairly general formulation of the two parties' risk-sharing problem. An optimal sharing rule $x^*(\cdot)$ is characterized by the following:

$$(1) \qquad x^*(\cdot) \in \underset{x(\cdot)}{\operatorname{argmax}}\ \mathcal{G}[x(\cdot)] \qquad \text{subject to} \qquad x(\cdot) \in \mathcal{F},$$

where

$$(2) \quad \mathcal{G}[x(\cdot)] \equiv G\left\{\int_{\mathcal{S}} U_1[x(s);s]\, d\mu(s);\quad \int_{\mathcal{S}} U_2[x(s);s]\, d\mu(s)\right\},$$

and $G : \mathbf{R}^2 \to \mathbf{R}$ is a general function that may take different characterizations depending on the bargaining process which leads to the risk-sharing agreement. In particular, such function may easily accommodate the standard Nash bargaining approach. The set $\mathcal{F}$, on the other hand, is a general feasible set that may incorporate restrictions such as $x(s) \geq 0$, for every $s \in \mathcal{S}$ or other technical constraints which guarantee the existence of a solution. Further, for given $U_i(\cdot;\cdot)$ and $\mu(\cdot)$ the constraint $x(s) \in \mathcal{F}$ could be interpreted as the constraint $\int_{\mathcal{S}} U_i[x(s);s]\, d\mu(s) \geq \overline{U}$ which allows us to reinterpret problem (1) as the standard coinsurance problem:

$$(3) \quad \max_{x(\cdot)} \int_{\mathcal{S}} U_i[x(s);s]\, d\mu(s)$$

$$\text{subject to} \qquad \int_{\mathcal{S}} U_j[x(s);s]\, d\mu(s) \geq \overline{U} \qquad j \neq i,$$

where $\overline{U}$ is the minimal level of utility that induces party j to accept the contract $x(\cdot)$.

A solution to problem (1) is called *first best*. We will denote the set of solutions to problem (1) by X^*, with typical element $x^*(\cdot)$. These first-best sharing rules are the benchmarks of our analysis. We shall take these benchmarks, hence problem (1), to be "well behaved" and as "simple" and possible. First, we assume that X^* contains only one element.

ASSUMPTION 1. The utility functions $U_i(\cdot;\cdot),(i = 1,2)$, the probability measure $\mu(\cdot)$, the function $\mathcal{G}(\cdot)$, and the feasible set $\mathcal{F}$ are such that a solution to problem (1) exists unique (up to a μ-measure zero set of states).

Second, we assume that the maximand function $\mathcal{G}(\cdot)$ satisfies the following mild continuity properties.

ASSUMPTION 2. The function $G : \mathbf{R}^2 \to \mathbf{R}$ defined as in equation (2) is such that given any sequence $\{(U_{1n};U_{2n})\}_{n=0}^{\infty}$ converging to $(U_1^*;U_2^*)$, we have that the sequence $\{G(U_{1n};U_{2n})\}_{n=0}^{\infty}$ converges to $\mathcal{G}[x^*(\cdot)]$

A substantial part of our analysis focuses on sharing rules that take only a finite number of (finite) values over $\mathcal{S}$. These are finite *step functions*. It will be convenient to assume that the feasible set $\mathcal{F}$ is sufficiently rich as to contain all finite step functions "sufficiently near" the first best.

ASSUMPTION 3. There exists a positive real number ϵ such that if $x(\cdot)$ is a finite step function and $\mathcal{G}[x^*(\cdot)] - \mathcal{G}[x(\cdot)] < \epsilon$ then $x(\cdot) \in \mathcal{F}$.[7]

II.2. Written Contracts as Computable Sharing Rules

It is widely agreed that the notion of *algorithmic* function captured by Turing computability is the widest possible one. We take the view that a good way to formalize the restrictions imposed on a contract by the fact that it must be possible to write it down, is to require that it should be algorithmic in nature. Intuitively, a written contract is a *finite* set of clauses that, given a realization of the state of nature, yield an outcome in a finite number of "steps." In our context, a finite number of steps can be interpreted as the fact that, examining the contract for a given realized state of nature it yields an outcome in finite time. It is easy to imagine a

7. If $\mathcal{F}$ contains only functions with, say, $x(s) \geq 0$ for all $s \in \mathcal{S}$, then this assumption should be modified to read if $x(\cdot)$ is a finite step function satisfying $x(s) \geq 0$ and $\mathcal{G}[x^*(\cdot)] - \mathcal{G}[x(\cdot)] < \epsilon$, then $x(\cdot) \in \mathcal{F}$.

written contract that, for instance, "loops" in the sense that clause α calls on clause β and vice versa. We exclude such contracts by assumption. It should also be emphasized that the notion of algorithmic which we employ to formalize the nature of written contracts may in fact be too wide. It is quite possible that further *legal* restrictions apply to what can be written into a contract or not. Since all our results hold for any class of functions that is contained in the class of Turing computable functions, these further restrictions do not affect our analysis below.

Before proceeding further, we introduce some notation concerning Turing machines. A Turing machine is identified by its *program*. A program is a finite string of symbols obeying some syntactical rules that we shall not specify here.[8] It follows that Turing machines, and hence computable functions, can be put in a one-to-one (computable) correspondence with the natural numbers.[9] This is a standard technique known as *Gödel numbering* (see, for instance, Cutland [1980] or for a brief exposition Anderlini [1989]). In the case of a finite set of symbols only being available, the numbering procedure can be intuitively thought of as assigning an order to the symbols to start with and then ordering the strings "alphabetically." To each string then there corresponds a number given by its place in the *dictionary* of all finite strings. We shall simply identify each machine with its Gödel number. Throughout the paper we will use the notation $\{x\}(s)$ to indicate the result of the computation of Turing machine $x \in \mathbf{N}$ with input $s \in \mathscr{S}$. It should be noted that the computation of a Turing machine on a given input may not *halt*. Intuitively the computation may loop. This will become of interest only in Section V below.

II.3. Computable Contracts

We now describe intuitively the "mechanics" of our written computable contracts. The formal details are lengthy, and hence they are relegated to the Appendix.

The "input" of the contract is given by the realized state of nature s. We consider the binary expansion of the real number s, which can always be thought of as a potentially infinite sequence of zeros and ones. Care should be taken since some real numbers

8. Any text on computability will give examples of such rules; see, for, instance, Cutland [1980].

9. This of course immediately proves that there are noncomputable functions since there are $2^{\aleph_0}$ functions from $\mathbf{N}$ into $\mathbf{N}$ but only countably many are in fact computable.

allow more than one binary expansion; in such case we assume that the expansion containing the largest number of *zero* digits is used. This infinite sequence can be thought of as the potentially available "evidence" or information about the state of nature s. This parallel is even more convincing if we interpret—as we do throughout the rest of the paper—the binary expansion of s in the following way. The realized state can be described by a countably infinite number of *characteristics* each of which can either be present or not in the description of the state of nature in question. The binary expansion of s can then be thought of as the (potentially infinite) ordered list of *yes* (1) or *no* (0) describing which characteristics are present in the description of a particular state of nature. Note that the restriction that the procedure embodied in the contract has to deliver an answer in finite time implies, for example, that only a finite amount of "evidential data" may be used, although "how much" of it will be used depends on the contract itself and potentially on the realized state of nature as well.

Given the input s, the contract can then be thought of as consisting of two distinct procedures: an *information-gathering* procedure that given the evidence delivers the finite amount of evidential data to be used by the contract, and an *outcome procedure* that actually determines the value of the sharing rule.

The information-gathering procedure can be described as follows. We start with no information. The procedure then specifies where to look first—in other words, which characteristics (or digits) of the available evidence to scan first. On the basis of the result of the first round of scanning, the information-gathering procedure tells us which set of digits to look at next, and so on for any finite number of rounds. We require the information-gathering procedure to be computable in the sense that the procedure itself can be embodied in a Turing machine, and, of course, to halt after a finite number of rounds. The set of all digits actually scanned (and their "positions") in the expansion of s is the evidential data gathered when the state of nature is s and constitutes the input of the outcome procedure. This part of the contract simply "computes" the value of the sharing rule as a function of the output of the information-gathering procedure. Since the outcome procedure is required to be computable, the value of the sharing rule that it can generate is itself described by a finite string of digits. We take the result of the contract to be a *rational* number with a finite number of nonzero digits. We will refer to such numbers as "regular" rational numbers, denoted by $\mathscr{Q}$ throughout the rest of the paper.

It is appealing to interpret the two components of our written contracts as the two main phases of the decision process of an enforcing agency such as a court. The "trial" starts with an information-gathering phase aimed at revealing to the court the available evidence. In a finite amount of time this phase is followed by the decision phase that delivers, again in finite time, the final decision of the court on what the contract actually prescribes corresponding to the realized state of nature.

It turns out that the set of sharing rules which can be thought of as an information-gathering and outcome procedure pair is equivalent to the set of sharing rules which can be computed by a Turing machine. Since the two definitions of a written contract are equivalent, we will use them interchangeably according to analytical convenience throughout the paper. The two formal definitions are as follows. (Their equivalence is proved in the Appendix.)

DEFINITION 1. A computable contract is a pair (f,g), where f is a computable information-gathering procedure yielding, for any given s, a finite set of digits with their "positions" in the binary expansion of s, and g is a computable outcome procedure yielding an outcome that is a regular rational number $c \in \mathscr{Q}$.

DEFINITION 2. A computable contract is a ("two-tape") Turing machine, whose Gödel number will be denoted by $x \in \mathbf{N}$ throughout the paper. The machine x is assumed to be such that for every $s \in \mathscr{S}$ its output $c = \{x\}(s)$ is defined and $c \in \mathscr{Q}$.[10]

II.4. Properties of Computable Sharing Rules

We are now in a position to partially characterize the properties of a computable contract. Intuitively, computability constrains the parties to sharing rules that are step functions. Further, it must be possible to "describe" the endpoints of these step functions with regular rational numbers. In what follows, it is much easier to think of computable contracts as a pair of computable functions (f,g) satisfying Definition 1.

Recall that we require the information-gathering procedure to halt after a finite number of rounds. Hence, the information-

10. Directly from Definition 2 we learn that constraining the parties to choose a computable contract reduces the cardinality of their feasible set of contracts to be countably infinite.

gathering procedure yields a *finite* subset of the digits of s for any $s \in \mathscr{S}$. Intuitively, one would then expect that, for any s in the interval comprising all $s \in \mathscr{S}$ which have identical digits in the first n positions, a given contract should yield a constant value. Hence, one would expect a computable contract to partition $\mathscr{S}$ into well-behaved intervals. Since whenever more than one binary expansion is available for a real number we choose the one with the largest number of zeros, one would also expect the intervals to be closed below but open above. This intuition turns out to be correct. We formalize this statement in Lemma 1 below. Some more notation is needed first.

Consider the set of pairs generated by the information-gathering procedure of a computable contract $x \in \mathbf{N}$ for any given state of nature s. Typically such a set takes the form,

$$f(s) = \{(s_{n_1},n_1),(s_{n_2},n_2), \ldots, (s_{n_k},n_k)\}, \tag{4}$$

where $s_{n_i} \in \{0,1\}$ is the n_ith digit of the binary expansion of s. Given $f(s)$ we define

$$n(\{x\},s) \equiv \max_k \{n_k \in f(s)\}.$$

In other words, $n(\{x\},s)$ is the right-most position in the binary expansion of s contained in $f(s)$. Further, we define $\phi(\{x\},s)$ as the real number that has as binary expansion the string of the first $n(\{x\},s)$ digits of s and zeros thereafter. Last, for any computable contract $x \in \mathbf{N}$ and any state of nature s, let $\Phi(\{x\};s)$ represent the following half-open (except if the upper limit is 1) interval:

$$\Phi(\{x\};s) \equiv [\phi(\{x\},s),\phi(\{x\},s) + 2^{-n(\{x\},s)-1}] \cap [0,1].$$

It is now possible to formalize our claim, proved in the Appendix, that a computable contract must partition $\mathscr{S}$ into well-behaved half-open intervals.

LEMMA 1. Every computable contract $\{x\}(\cdot)$ takes the form of a step function that partitions $\mathscr{S}$ into a collection of disjoint half-open intervals having regular rational endpoints. In other words, given any computable contract $\{x\}(\cdot)$, we have

(i) For every $s \in \mathscr{S}$ and every $s' \in \mathscr{S}$, $s' \in \Phi(\{x\};s)$ implies that $\{x\}(s') = \{x\}(s)$.

(ii) The interval $\Phi(\{x\};s)$ is well defined for every $s \in \mathscr{S}$.

(iii) For very $s \in \mathscr{S}$ and $s' \in \mathscr{S}$, either $\Phi(\{x\};s) = \Phi(\{x\};s')$, or $\Phi(\{x\};s) \cap \Phi(\{x\};s') = \varnothing$.

A partial converse of Lemma 1 holds. Lemma 2 below states that given any *finite* collection of disjoint half-open intervals with regular rational numbers as endpoints and any corresponding set of regular rational values, there exists a computable contract which takes precisely these values on each of the intervals. This is the key to the approximation result discussed in Section IV below.

LEMMA 2. Given any ordered finite set of regular rational numbers in $\mathcal{S}$: $s_0 < s_1 < \cdots < s_n$, $(s_0 = 0, s_n = 1)$ and any finite set of regular rational values, $v_i \in \mathcal{Q}$, $(i = 1, \ldots, n)$, there exists a computable contract $x \in \mathbf{N}$ such that

$$s \in (s_{i-1}, s_i) \Rightarrow \{x\}(s) = v_i \qquad \forall i = 1, \ldots, n.$$

Proof of Lemma 2. Let $\hat{n}$ be the position of the right-most 1 in the binary expansion of any s_i. Ensure now that the function f corresponding to x is such that given any $s \in \mathcal{S}$ exactly the first $\hat{n}$ digits of s are scanned. On the basis of this scanning it is always possible to determine, for any state of nature s, which is the index $i = 1, \ldots, n$ such that $s \in (s_{i-1}, s_i)$. Hence, since the constructed set of half-open intervals is *finite,* a g function that attributes the corresponding value v_i to each interval is certainly computable.

QED

We conclude this section by noting that it is easy to construct extremely simple examples of first-best sharing rules that cannot be implemented by a computable contract. Consider, for instance, a familiar coinsurance problem whose first-best sharing rule takes the value $l > 0$ for any state of nature s on the unit interval except for the state $\bar{s}$, corresponding to which it takes a strictly smaller value h:

$$x(s) = \begin{cases} h & \text{if } s = \bar{s} \\ l & \text{otherwise,} \end{cases} \tag{5}$$

where $h < l$. This sharing rule is clearly not computable by Lemma 1.

III. CONTRACT INCOMPLETENESS

III.1. A Definition

In order to test whether the computability restrictions we impose on contracts yield incomplete contracts, we need a formal definition of what an incomplete contract is.

We take the view that a satisfactory definition should apply to any type of contract, computable or not, and should capture the intuitive notion that a contract is incomplete if it neglects information about states of nature which would be optimal for the contracting parties to include.

Some examples are the best way to rule out some possible definitions that may seem natural at the outset. Consider the case of a simple coinsurance problem yielding a first-best sharing rule as in (5). One would want to call incomplete a contract of the type,

$$x(s) = \begin{cases} h' & \text{if } s' < s < s'' \\ l' & \text{otherwise} \end{cases} \tag{6}$$

with $s' < \bar{s} < s''$ amd $h' < l'$. This is because it is intuitively appealing to say that in such a contract the "accident" state $\bar{s}$ has been "incompletely" or "loosely" defined. It is clear, however, that the partition of $\mathscr{S}$ induced by such $x(\cdot)$ is neither coarser nor finer than the partition induced by the first-best contract.[11] We conclude that it would not be satisfactory to define incomplete contracts simply as contracts that "induce a partition of the state space which is strictly coarser than the one induced by the first best."

Concrete examples of contracts that use "coarse" clauses to define loosely a set of states which is payoff relevant seem to be common in practice. Consider, for instance, the common clause guaranteeing compensation against "pain and suffering." A contract prescribing payment in case of pain and suffering can be seen as using a loose definition of a set of states that while quite precisely defined in the minds of the contracting parties is hard to describe precisely in a written contract. The contract is incomplete in the sense that it incompletely describes the relevant set of states by use of the pain and suffering formula. Another example of contractual statements that only represent approximate and incomplete descriptions of some relevant set of states is the "act of God" clause that, for instance, dispenses airlines from their obligation to supply the agreed-upon services. Clearly, the exact description of the events that are an act of God is missing, but the set of possible states of nature that fits the definition, however, is restricted by the statement.

11. Notice that in the example above the fact that the accident state is characterized by a unique state of nature is completely inessential. In fact, a similar example may be constructed in which the event accident is defined by a set, for example, a whole interval, of states of nature $[a,b]$.

A second difficulty would arise if we defined as incomplete only those contracts that partition the state space in a way coarser than the first best. Consider the standard example in which the first best requires the parties to write a sharing rule contingent on two variables. Any contract that specifies a sharing rule contingent only on one of the two variables should reasonably be called incomplete. However, in this case again the partition of the (two-dimensional) state space that the single variable contract induces may well be neither coarser nor finer than the partition induced by the first-best contract.

In view of the difficulties we have outlined, we propose to call a contract incomplete if it partitions the state space in a manner that could *only* be optimal if the contract design were constrained by the parties' inability to distinguish among relevant states of nature. We start writing down this definition formally and then introduce a simpler but equivalent definition that we will use in the remainder of the paper.

First, we need some additional notation. Let Π be the set of all possible partitions of the state space $\mathcal{S}$.[12] Let, $\mathcal{P} \in \Pi$ be a partition of the state space $\mathcal{S}$, and $\mathcal{I}(\mathcal{P}, s)$ the element of the partition $\mathcal{P}$ to which a given state of nature s belongs. We denote by $\overline{\mathcal{P}}$ the partition of the state space whose elements are the singleton of each state of nature so that $\mathcal{I}(\overline{\mathcal{P}}, s) = \{s\}$ for every $s \in \mathcal{S}$. Define now $P(x) \in \Pi$ as the partition of the state space $\mathcal{S}$ induced by the contract $x(\cdot)$. Formally, $P(x)$ is defined by

$$(7) \qquad \mathcal{I}(P(x), s) = \{s' \in \mathcal{S} | x(s') = x(s)\}.$$

We use the notation $\mathcal{P} \geq \mathcal{P}'$ to mean that $\mathcal{P}$ is equal to or coarser than $\mathcal{P}'$, or equivalently $\mathcal{P}'$ is equal to or finer than $\mathcal{P}$.

Consider the following version of problem (1), where the sharing rules the parties may choose vary only across different elements of the partition $\mathcal{P}$:

$$(8) \quad \max_{x(\cdot)} G\left\{\int_{\mathcal{S}} U_1[x(\mathcal{I}(\mathcal{P}, s)); s]\, d\mu(s); \quad \int_{\mathcal{S}} U_2[x(\mathcal{I}(\mathcal{P}, s)); s]\, d\mu(s)\right\}$$
$$\text{subject to} \qquad x(\cdot) \in \mathcal{F}.$$

Let $X(\mathcal{P})$ be the set of sharing rules $x(\cdot)$ that solve problem (8). By construction, every $x(\cdot) \in X(\mathcal{P})$ must be invariant with respect to states of nature that belong to the same element of $\mathcal{P}$. Therefore, the solution(s) to problem (8) can be viewed as the contract(s) that the parties would choose if they were *constrained* not to distin-

12. We only consider partitions whose elements are Lebesgue-measureable sets.

guish between states which belong to the same cell of the partition $\mathscr{P}$. The parties are *informationally* constrained by $\mathscr{P}$. This constraint clearly "vanishes" if in (8) we set $\mathscr{P} = \overline{\mathscr{P}}$. In other words, $X(\overline{\mathscr{P}})$ coincides with the set of first-best sharing rules X^*. We are now ready for a formal definition of incomplete contracts.

DEFINITION 3. A contract $x(\cdot)$ is *incomplete* if and only if

$$\overline{\mathscr{P}} \notin \{\mathscr{P} \mid \exists \hat{x} \in X(\mathscr{P}) \text{ such that } P(\hat{x}) \geq P(x)\}. \tag{9}$$

In other words, a contract is incomplete if and only if the partition of the state space it induces (or any coarsening of it) can only be the result of an optimizing choice of contract in which the parties are informationally constrained and these constraints are *binding*. Less information about states of nature is included in an incomplete contract than what would be optimal if there were no constraints.

Definition 3 is cumbersome to use as it stands. It is, however, equivalent to a much simpler formal statement. Let $\Pi^* \subseteq \Pi$ (with typical element $\mathscr{P}^*$) be the class of partitions that are equal to or finer than any partition of the state space induced by a first-best sharing rule. In other words,

$$\Pi^* \equiv \{\mathscr{P} \in \Pi \mid \exists x^* \in X^* \text{ such that } P(x^*) \geq \mathscr{P}\}.$$

Consider now the following alternative definition of incomplete contracts.

DEFINITION 4. A contract $x(\cdot)$ is *incomplete* if and only if

$$P(x) \notin \Pi^*. \tag{10}$$

In other words, a contract is incomplete if and only if it induces a partition of the state space that is not as fine as or finer than the partition induced by any first-best contract. It is not hard to show that the two definitions we have given coincide.

LEMMA 3. Definitions 3 and 4 are equivalent.

Proof of Lemma 3. Assume that $x(\cdot)$ is complete according to Definition 4. Therefore, $P(x) \in \Pi^*$. Since $X^* = X(\overline{\mathscr{P}})$, the definition of Π^* directly implies that $x(\cdot)$ is complete according to Definition 3. Assume now that $x(\cdot)$ is complete according to Definition 3. Then (again using the fact that $X^* = X(\overline{\mathscr{P}})$), there exists an $x^* \in X^*$ such that $P(x^*) \geq P(x)$, which implies that $P(x) \in \Pi^*$ or equivalently $x(\cdot)$ is complete according to Definition 4.

QED

It is important to notice that the incompleteness of a contract as in Definitions 3 or 4 is related to but is distinct from the optimality of the contract. This is not surprising since we have defined incompleteness as a feature of the *partition* of the state space that a contract induces while optimality is clearly a feature of the entire sharing rule, including the values it takes.

LEMMA 4. If a contract $x(\cdot)$ is incomplete, then it is suboptimal in the sense that $\mathcal{G}[x(\cdot)] < \mathcal{G}[x^*(\cdot)]$, while the reverse implication is not true.

Proof of Lemma 4. The fact that an incomplete contract must be suboptimal is an immediate consequence of Definition 4. Hence, we omit the details. To prove that the reverse implication is not true, an example will suffice. Consider, for instance, a problem yielding a simple coinsurance first-best rule as in (5). Observe next that any sharing rule $x(\cdot)$ which is, say, strictly increasing over $\mathcal{S}$ is complete since the partition it induces consists of all singleton sets. Since such a sharing rule is clearly suboptimal for a simple coinsurance problem, this is enough to prove the claim.

QED

We conclude by noting that Definition 4 fits the notion of incompleteness for the example of a simple coinsurance problem which we mentioned at the beginning of this section. Indeed, it is immediate to check that, given the first-best sharing rule described in (5), the contract described in (6) is incomplete according to Definition 4. Similarly, a contract contingent on only one of two variables on which the first-best contract depends is incomplete according to the definition we have proposed.

III.2. Silent Contracts

As we mentioned in the Introduction, the definition of incompleteness we are proposing differs from that of silent contracts most often used in the literature. In our view, however, the two are not in contrast but on the contrary intimately related.

Consider again a silent contract *together* with the ex post mechanism that is used in the uncontracted contingencies. We shall assume, as in the literature, that the parties are aware of the ex post mechanism and hence that the two together can in a sense be viewed as a particular contract which does make prescriptions for *all* states of nature. The ex post mechanism may constrain the resulting contract in two distinct ways. First, it may constrain the *values* of the sharing rule on the uncontracted contingencies, for instance, because of preassigned property rights. Second, it may not be possible to fine tune the outcome of the ex post mechanism so as to make its prescription vary sufficiently with the realized state of nature. Take, for instance, the extreme case in which the ex post mechanism must give the same outcome for all uncontracted contingencies, but the actual outcome can be chosen in an ex ante optimal manner. The analogy with problem (8) is apparent. The parties can vary the sharing rule as they please on the set of contracted contingencies, but not at all on the uncontracted ones. The solution to problem (8) with the appropriate informational constraint is equivalent to the optimal silent contract plus the ex post mechanism. In essence, if we view the crucial problem posed by silent contracts to be the fact that the ex post mechanism cannot be made to vary at will with the realized state (rather than the fact that the set of possible outcomes may be restricted by the ex post mechanism), then the two definitions are very close indeed. Whenever the inflexibility of the ex post mechanism is a *binding* constraint for a silent contract, then that contract plus the ex post mechanism is equivalent to a contract which does make a prescription for every state of nature and which is incomplete according to the definition we have proposed.

If we apply the definition of incompleteness proposed here to the equilibrium contracts plus the ex post mechanism, obtained in Grossman and Hart (1986) and Hart and Moore [1988, 1990], we find that they are indeed incomplete. Conversely, the equilibrium contracts plus the ex post mechanism derived when renegotiation design is feasible [Chung 1991; Aghion, Dewatripont, and Rey 1994] or when the court's information structure is rich enough [Nöldeke and Schmidt 1994] are instead complete: the partition they induce on the domain of the contract coincides with the first-best one.

IV. The Approximation Result

The algorithmic nature of contracts—modeled as Turing machines—is not enough by itself to derive equilibrium incomplete contracts. Given the natural continuity properties and boundedness of the parties' expected utility in their share of the surplus, while in some cases the parties will be unable to write the first-best contract, they will always succeed in approximating it, in terms of their expected utilities, by means of a written contract. Thus, it is hard to argue that, without any additional constraint, the algorithmic nature of contracts explains the pervasive incompleteness which we observe.

We formalize this *approximation result* in the following proposition.

PROPOSITION 1. Given any $\xi > 0$, there exists a feasible computable contract $\{x\}(\cdot) \in \mathbf{N}$ such that

$$\mathcal{G}[x^*(\cdot)] - \mathcal{G}[\{x\}(\cdot)] < \xi. \tag{11}$$

The formal proof of Proposition 1 is in the Appendix. An intuitive outline of the argument is quite simple, however.

By standard results it is always possible to decompose the probability measure μ on the state space $\mathcal{S}$ into the weighted sum of a purely atomic measure α and a nonatomic measure γ. Hence, the problem can be divided into two parts.

As far as the purely atomic measure is concerned, we can proceed as follows. If the purely atomic measure α has an infinite number of atoms, we can always choose a finite but sufficiently large n such that the first n atoms of α have a total probability arbitrarily close to one. If α has a finite number of atoms, we consider all of them. In any case we can restrict attention to a finite number of atoms. Around each element of such a finite set of atoms, we can then construct a small half-open interval so that the probability of the entire collection of half-open intervals has arbitrarily small probability according to the nonatomic measure γ. We can then define a finite step function that takes regular rational values arbitrarily close to the values of the first-best $x^*(\cdot)$ on the atoms of μ, on the small intervals we have constructed around each atom, and is zero otherwise.

We then turn to the nonatomic component γ of the original measure μ. Since the first best $x^*(\cdot)$ is a Lebesgue-measurable function (otherwise expected utility would not be defined), we know that the first-best expected utility when the probability measure is γ can be approximated arbitrarily closely using a finite step function instead of $x^*(\cdot)$ itself. This implies that the same approximation can also be carried out with a finite step function which partitions $\mathcal{S}$ into a finite collection of half-open intervals with regular rational endpoints and taking regular rational values.

Finally, we combine the two finite step functions we have constructed for the purely atomic and the nonatomic components of μ into a single finite step function as follows. The single step function is set equal to the one we obtained for the purely atomic measure α for all states in the small intervals around the atoms, and equal to the step function we have obtained for the nonatomic measure γ for the rest of the state space. Since the utility functions of both parties are continuous in x and bounded, with this construction we can clearly approximate the first-best utility level for both parties by a sharing rule that is a finite step function with regular rational endpoints and taking regular rational values. By Lemma 2 this implies that this approximation can be carried out using a computable sharing rule $\{x\}(\cdot)$. Since we have assumed that $\mathscr{F}$ is sufficiently rich as to contain all step functions near the first best (Assumption 2), $\{x\}(\cdot)$ must be feasible. Since $\mathscr{G}(\cdot)$ is continuous near the first best (Assumption 3), this clearly implies the claim.

The approximation result shows that the level of expected utility which the parties achieve in the first best can always be approximated by a computable sharing rule. Suppose now that the first-best sharing rule is not itself computable. Then the problem of finding an optimal *computable* contract will not have a solution. In fact, the first-best expected utility of the parties can be approximated arbitrarily closely, but not achieved by a computable contract. The following remark identifies which conditions guarantee that an optimal computable contract may be found.

REMARK. Suppose that Assumptions 1, 2, and 3 hold. Consider problem (1) with the additional constraint that x is a computable contract:

$$\max_{\{x\}(\cdot)} \mathscr{G}[\{x\}(\cdot)] \quad \text{subject to} \quad \{x\}(\cdot) \in \mathscr{F} \quad \text{and} \quad x \in \mathbf{N}. \tag{12}$$

Then problem (12) has a solution if and only if the set X^* of solutions of problem (1) contains at least one computable sharing rule.

V. Endogenously Incomplete Contracts

This section presents the main result of the paper. Before getting into further formal details, it is necessary to review our findings so far and to outline the direction we are about to take. In Section II and in particular in subsection II.4, we have introduced and characterized the behavior of contracts that are restricted to be *algorithmic* or *computable* in the Turing sense. We argued that such restriction on the sharing rule identifying a contract is an appropriate way to model the *formal* nature of written contracts. It turns out that it is not difficult to show that some contracts cannot be written formally. This is, roughly speaking, just because of the basic mathematical fact that not all functions are computable.

In Section IV, on the other hand, we found that without further structure on the problem of finding an *optimal* contract the computability restriction has little impact on the outcome of the maximization process leading to the choice of a contract. In particular, when the first-best sharing rule is not itself computable, the maximization problem leading to the choice of an optimal contract will not have a solution. However, a sequence of computable contracts can be found that approximates more and more closely the first best. Thus, without further restrictions, the parties will always *approximate* the choice of a *complete* contract.

In this section we introduce further restrictions on the maximization process leading to the choice of an optimal contract which in turn lead us to conclude that for some first-best sharing rules the parties will in fact select an incomplete contract. In other words, we will find genuinely endogenously incomplete contracts. The main restriction we impose is that the choice process which leads to the selection of a particular contract should itself be computable in some appropriate sense. We have in mind two different, and not necessarily alternative, motivations to this restriction. According to the first motivation, the formality of the legal system imposes on contracting parties an additional technological constraint. The bargaining procedure that leads the parties to the selection of a computable contract is so structured, by legal rules or common practice, as to require parties, or their lawyers, to present *formal* arguments to support their choice between any two computable contracts. These formal arguments yield a choice criterion that is itself formal, hence algorithmic. An example of this type of restrictions can be envisaged in the rules that discipline contracting between Government agencies and private firms: any proposal needs to be submitted and replied to in a formal manner. A more direct—but probably less palatable to some readers—way to motivate this restriction is to think of rational agents as Turing machines [Binmore 1987; Anderlini 1989; Canning 1992]. Clearly, this way of modeling agents leads to a contract selection process that is computable.

Intuitively, what drives the result we derive in this section is the following. The formality of the choice process leads us in a natural way to conclude that whenever the parties try to approximate a given noncomputable first best they will do so through a sequence of computable contracts which is itself computable as a sequence. We then find that for some contracting problems the approximating sequences of Proposition 1 are not computable. Therefore, we are able to conclude that in some cases, the parties will select a contract that is suboptimal in a way that makes it incomplete.

V.1. Preliminaries

We start by introducing the formal definition of the choice criterion through which contracting parties are assumed to identify their preferred computable contract.

DEFINITION 5. A choice criterion $C \in \mathbf{N}$ is a Turing machine such that

$$\{C\}(x,x') = \begin{cases} x & \text{only if } \mathscr{G}(x) \geq \mathscr{G}(x'); \\ x' & \text{only if } \mathscr{G}(x) < \mathscr{G}(x'). \end{cases} \tag{13}$$

It is important to notice that Definition 5 does *not* stipulate that a choice criterion should give an answer for any pair of contracts. It only requires a choice criterion to conform to the ranking implicit in $\mathscr{G}(\cdot)$ whenever an answer is given. In other words, for some pairs (x,x') the computation $\{C\}(x,x')$ could generate a "meaningless" output, equal to neither x nor x', or it could simply loop and not halt.

We view a choice criterion $C \in \mathbf{N}$ as embodying the process of contract selection in the following sense. Starting with a particular contract x the choice criterion is used to "look" for a better one if this can be found. Once a better contract is found, the criterion is used again to see whether a new improving contract can be found and so on, possibly ad infinitum, thus generating a sequence of contracts which is weakly monotonic in terms of $\mathscr{G}(\cdot)$. In the first place, we would like the choice process generated by any choice criterion $\{C\}(\cdot,\cdot)$ to be independent of the initial computable contract supplied as input to $\{C\}(\cdot,\cdot)$. Hence, we restrict attention to criteria $\{C\}(\cdot,\cdot)$ that are *pseudo-complete* in the following sense. For any choice criterion $C \in \mathbf{N}$ define

$$\mathscr{R}_C \equiv \{x \in \mathbf{N} \mid \exists x' \in \mathbf{N} \text{ such that } \{C\}(x,x') = x, \text{ or } x'\} \tag{14}$$

as the set of computable contracts for which $\{C\}(\cdot,\cdot)$ gives a meaningful answer.

DEFINITION 6. A choice criterion $C \in \mathbf{N}$ is pseudo-complete if and only if $\mathscr{R}_C$ is not empty and for every pair $(x,x') \in \mathscr{R}_C^2$:

$$\{C\}(x,x') = \begin{cases} x & \text{if and only if } \mathscr{G}(x) \geq \mathscr{G}(x'); \\ x' & \text{if and only if } \mathscr{G}(x) < \mathscr{G}(x'). \end{cases}$$

We are now in a position to associate to any choice criterion $\{C\}(\cdot,\cdot)$ which is pseudo-complete a sequence of computable contracts. For any given computable contract $x_{n-1} \in \mathscr{R}_C$, select the next element of the sequence, x_n, as the computable contract with the lowest Gödel number that guarantees $\{C\}(x_n,x_{n-1}) = x_n$.[13] Formally, we associate to any criterion C a "choice sequence" as follows:

REMARK. To each choice criterion $C \in \mathbf{N}$ is associated a (unique) choice sequence of computable contracts $\{\{x_{n,C}\}(\cdot)\}_{n=0}^{\infty}$ as follows:

$$x_0 \equiv \min \{x \mid x \in \mathscr{R}_C\}$$

and

$$(15) \quad x_n \equiv \begin{cases} \min \{x \mid x \in \mathscr{R}_C, \{C\}(x,x_{n-1}) = x, x \neq x_i, & \text{if such } x \text{ exists } \forall i \leq n-1\} \\ x_{n-1} & \text{otherwise.} \end{cases}$$

Two things should be noted about the choice sequence associated with any pseudo-complete C. The first is that even if the set $\mathscr{R}_C$ is finite, we still associate an *infinite* sequence $\{\{x_{n,C}\}(\cdot)\}_{n=0}^{\infty}$ with each choice criterion pseudo-complete C, simply by repeating ad infinitum the "last" element of $\mathscr{R}_C$. This is purely for simplicity in what follows. The second is that the sequence (except for a finite initial set of elements) does not depend on the particular initial value x_o because of the pseudo-completeness of C.

If the class of computable contracts $\mathscr{R}_C$ is *finite*, there exists, of course, no difficulty in defining the computable contract to which the choice sequence $\{\{x_{n,C}\}(\cdot)\}_{n=0}^{\infty}$ converges. On the other hand, if $\mathscr{R}_C$ is *infinite*, the problem becomes more complex. In fact, by construction, we know that the sequence, being monotonic and bounded, converges in $\mathscr{G}(\cdot)$ terms. Unfortunately, this does not imply convergence of the choice sequence in any way that is meaningful as far as the completeness or incompleteness of the

13. Note that the choice to minimize the Gödel number of the computable contract is completely arbitrary, any other mechanism will do. In fact, Gödel numbers themselves may be reassigned using any computable bijection $f: \mathbf{N} \to \mathbf{N}$.

limit contract is concerned. We focus on the properties of the limit contract of a potentially infinite sequence since, as Proposition 1 shows, this corresponds to the least likely situations in which incompleteness may arise. Hence, we restrict attention to a set of criteria $C \in \mathbf{N}$ which allows us meaningful statements about the limit completeness or incompleteness of their associated choice sequence. Since incompleteness is a property of the induced partitions, we would like the choice criterion to select a sequence of computable contracts such that the sequence of induced partitions of the state space converges itself in an appropriate sense. Therefore, we need to define a *metric* on the set of partitions of the state space $\mathcal{S}$.

DEFINITION 7. On the set Π of all partitions of $\mathcal{S}$, we define the following distance between two partitions $\mathcal{P} \in \Pi$ and $\mathcal{P}' \in \Pi$:

$$d(\mathcal{P},\mathcal{P}') = \sup_{s \in \mathcal{S}} h(\mathcal{I}(\mathcal{P},s);\mathcal{I}(\mathcal{P}',s)), \tag{16}$$

where $h(\cdot;\cdot)$ denotes the Hausdorff metric over sets.[14]

Since $h(\cdot;\cdot)$ is, by definition, a metric, we conclude that $d(\cdot,\cdot)$ is a metric as well.

We can now impose a second property on the choice criteria C to be considered.

DEFINITION 8. A choice criterion $C \in \mathbf{N}$ is convergent in partitions if and only if the associated sequence of computable contracts $\{\{x_{n,C}\}(\cdot)\}_{n=0}^{\infty}$ is such that the corresponding sequence of induced partitions $\{P(x_{n,C})\}_{n=0}^{\infty}$ converges in the metric on partitions of Definition 7.

We denote with $\mathcal{P}_C$ the partition to which the sequence of induced partitions converges.[15] Any criterion C which is pseudo-complete and convergent in partitions will allow us to meaningfully ask the question of whether the associated choice sequence converges to a partition of the state space which is associated with a contract that is complete or incomplete. As is clear from the proof of Lemma 4, however, it would be very simple to construct choice criteria C which produce a choice sequence that converges to a contract that

14. Let A and B be two subsets of the real line. The Hausdorff distance between A and B is defined as $h(A;B) \equiv \max\{\inf_{x \in A} \sup_{y \in B} e(x;y); \inf_{y \in B} \sup_{x \in A} e(x;y)\}$, where $e(x;y)$ is the Euclidean distance between x and y (see, for instance, Kelley [1969]).

15. Notice that the convergence in partition (Definition 8) is neither implied nor implies pointwise convergence of the computable sharing rules of the constructed sequence.

is complete, whatever the shape of the first best. An easy example is a choice sequence that converges pointwise to an everywhere strictly increasing sharing rule. Such criteria, however, are typically "wrong" on the grounds that they do not do a very good job at optimizing according to $\mathscr{G}(\cdot)$. Intuitively, such criteria, in the limit, use all the information about the realized state, but simply in the wrong way, relative to the first-best sharing rule. Intuitively, the next restriction we impose on the set of criteria we consider is the following. In the limit, the choice sequence should partition the state space more and more finely, *only if* this is advantageous in terms of the objective function $\mathscr{G}(\cdot)$. We call this property *efficient use of information.* One interpretation (which we do not formalize fully for reasons of space) of the following restriction is that of a lexicographic cost (to be minimized after $\mathscr{G}(\cdot)$ has been maximized) of partitioning more and more the state space. Formally, we state this property as

DEFINITION 9. A choice criterion $C \in \mathbf{N}$ makes efficient use of information if and only if, given $x \in \mathscr{R}_C$, whenever there exists an $x' \in \mathscr{F}$ computable such that $P(x') > P(x)$ and $\mathscr{G}(x') \geq \mathscr{G}(x)$, then $x' \in \mathscr{R}_C$.

In words, the previous definition says that a choice criterion makes efficient use of information if it is eventually bound to select a computable contract x' that induces a strictly coarser partition of the state space than a given contract x and dominates it, in $\mathscr{G}(\cdot)$ terms.

We call *sensible* the choice criteria $C \in \mathbf{N}$ which are *pseudo-complete,* are *convergent in partitions,* and *make efficient use of information.* In what follows, we will denote with the symbol $\mathscr{C} \subset \mathbf{N}$ the class of sensible criteria.

Since we have not made any assumption on the objective function $\mathscr{G}(\cdot)$, nor about the concavity of the utility functions U_i, Definition 9 is almost empty as it stands. Suppose, for instance, that $G(\cdot;\cdot)$ were, say, monotonic in the sum of its arguments, and that both parties were risk-lovers. Then, the assumption of efficient use of information would have very little impact on the characteristics of the possible choice sequence associated with a sensible criterion $C \in \mathbf{N}$. This is simply because, by making the value of the sharing rule "vary more and more" with the state, successive increments in $\mathscr{G}(\cdot)$ could be obtained. The last assumption we shall make is that problem (1) satisfies a form of "riskaversion." Since we are dealing with a state-dependent-utility formulation, this takes a slightly unfamiliar form.

Consider a sharing rule $x(\cdot)$ which induces, at least on some subset of the state space $\mathscr{S}$, a partition strictly finer than the one induced by some first-best sharing rule $x^*(\cdot)$. In other words, assume that $x(\cdot)$ is such that we can choose s_1 and s_2 in such a way that $\mathscr{I}(P(x),s_1) \cap \mathscr{I}(P(x),s_2) = \varnothing$ and $\mathscr{I}(P(x^*),s_1) = \mathscr{I}(P(x^*),s_2)$. From $x(\cdot)$ we can construct a sharing rule $x'(\cdot)$ "averaging" the values of $x(\cdot)$ in the following sense. Define $x'(\cdot)$ as follows:

$$(17) \qquad x'(s) \equiv \begin{cases} x(s) & \text{if } s \notin \mathscr{I}(P(x),s_1) \cup \mathscr{I}(P(x),s_2) \\ \bar{c} & \text{if } s \in \mathscr{I}(P(x),s_1) \cup \mathscr{I}(P(x),s_2). \end{cases}$$

ASSUMPTION 4. There exists a regular rational number $\bar{c} \in \mathscr{Q}$ such that $\mathscr{G}(x') \geq \mathscr{G}(x)$ and $x'(\cdot) \in \mathscr{F}$.

V.2. *Endogenously Incomplete Contracts: Preamble*

The spirit of the computability restrictions we have imposed on the sharing rule and on the choice criteria is that we are prepared to accept the assumption that they both should be algorithmic in the widest possible sense. This does not entail that we are, for instance, allowing specific bounds on the "complexity" (however defined) of either object. Both the contract and the arguments used for selecting it must be capable of being "formalized" in the widest possible sense of the word. Intuitively, we are examining the "limit case" in two specific senses. The contract technology simply restricts the parties to what can be written down, but not to any specific contract length (as measured by, say, the number of clauses or some other complexity measure). The criterion through which the contract selection is made, again, must be capable of being formalized, but no specific bounds on how "smart" such procedure can be are imposed. It is therefore clear that for some forms of problem (1) (or alternatively, for some 5-tuples $U_1(\cdot;\cdot)U_2(\cdot;\cdot)\mu(\cdot)G(\cdot;\cdot)$ and $\mathscr{F}$) it will be possible to find a criterion C such that the associated choice sequence converges to a complete contract. The simplest case is that of a computable first best. A less obvious situation is that of a contract problem which is such that the "approximating sequences" of Proposition 1 are computable in the sense that they are the choice sequences associated with some criterion C. Therefore, our results in the following two subsections all take the following general form. For some contract problems it is the case that for *any* sensible choice criterion $C \in \mathscr{C}$ the associated choice sequence converges to a partition which is associated with an incomplete contract.

We take this to mean that the restriction that both the sharing rule and the choice criterion be algorithmic in nature will, in appropriate cases, yield an endogenous *choice* of an incomplete contract. We partially characterize a specific class of problems yielding endogenously incomplete contracts in subsection V.4 below.

V.3. Endogenously Incomplete Contracts: Results

To demonstrate that genuinely endogenously incomplete contracts may occur, it is enough to show that such incomplete contracts arise in any class of contract problems. Hence, to keep things simple and without any loss of generality, we restrict attention to the class of simple coinsurance problems, characterized by a probability measure $\mu(\cdot)$ which has an atom at the state of nature $\bar{s}$ and is (say) uniform in the remaining support and yields a first-best sharing rule of the type defined in (5). We parameterize this class with the single "accident" state $\bar{s}$.

PROPOSITION 2. There exist contract problems—i.e., there exist simple coinsurance problems $\bar{s}$—such that for every sensible criterion $C \in \mathscr{C}$:

(i) the limit partition $\mathscr{P}_C$ is incomplete;
(ii) there exists an index $\bar{n}$ such that $x_{n,C}$ is incomplete for every $n \geq \bar{n}$.

Proof of Proposition 2. Recall from Lemma 1 that any computable contract $x \in \mathbf{N}$ partitions the state space $\mathscr{S}$ into a collection of half-open intervals. Hence, since $P(x_{n,C})$ consists of half-open intervals for every $n \in \mathbf{N}$ statement, (ii) follows since the first best must take the same form as (5). We still need to prove statement (i). We first prove that if for any given sensible $C \in \mathscr{C}$, $\mathscr{P}_C$ is complete, then it must be the case that $\mathscr{P}_C$ partitions $\mathscr{S}$ into the two sets $\{\bar{s};[0,\bar{s}) \cup (\bar{s},1]\}$. Denote by $l(\bar{s};x_{n,C})$ the half-open interval induced by $x_{n,C}$ around $\bar{s}$, $\bar{s} \in l(\bar{s};x_{n,C})$, and by $a(\bar{s};x_{n,C})$, $b(\bar{s};x_{n,C})$, respectively, the lower and upper bounds of $l(\bar{s};x_{n,C})$. Notice that since the sequence of induced partitions $\{P(x_{n,C})\}_{n=0}^{\infty}$ converges to $\mathscr{P}_C$ in the metric of Definition 7, $\mathscr{P}_C$ can only be complete if

$$\lim_{n\to\infty} a(\bar{s};x_{n,C}) = \lim_{n\to\infty} b(\bar{s};x_{nC}) = \bar{s}.$$

Suppose now that the sequence $\{P(x_{n,C})\}_{n=0}^{\infty}$ is such that for arbitrarily large n we have that $P(x_{n,C})$ partitions at least one of the two intervals $[0;a(\bar{s},x_{n,C})]$ and $[b(\bar{s};x_{n,C}),1]$ into two or more subsets. By Assumption 4, this implies that it is possible to find a computable contract $\hat{x}$ which coarsens $P(x_{n,c})$ into just $\{[a(\bar{s};x_{n,C}),b(\bar{s};x_{n,C}));$ $[0,a(\bar{s};x_{n,C})) \cup [b(\bar{s};x_{n,C}),1]\}$ and satisfies $\mathscr{G}(\hat{x}) \geq \mathscr{G}(x_{n,C})$. Hence, by efficient use of information we have $\hat{x} \in \mathscr{R}_C$ and, since choice sequences are always weakly monotonic in $\mathscr{G}(\cdot)$, there exists an arbitrarily large index m such that $\hat{x} = x_{m,C}$. Notice now that Definition 7 of the metric on partitions implies that

$$(18) \qquad d[P(\hat{x}),P(x_{n,C})] > \{\tfrac{1}{2}[a(\bar{s};x_{n,C}) + (1 - b(\bar{s};x_{n,C}))]\}.$$

Hence, for arbitrarily large n and m, $d[P(x_{m,C}),P(x_{n,C})]$ is bounded away from zero, which *contradicts* the hypothesis that the sequence of induced partitions $\{P(x_{n,C})\}_{n=0}^{\infty}$ converges to $\mathscr{P}_C$. Therefore, we have shown that if $\mathscr{P}_C$ is complete it necessarily takes the form $\{\bar{s};[0,\bar{s}) \cup (\bar{s},1]\}$.

Suppose now, by way of contradiction, that claim (i) is false. Then we would have that for every $\bar{s} \in \mathscr{S}$ there exists a $C \in \mathscr{C}$ such that $\mathscr{P}_C$ takes the form $\{\bar{s};[0,\bar{s}) \cup (\bar{s},1]\}$. The latter is a *contradiction* since there are uncountably many possible $\bar{s}$, while there are only countably many possible $C \in \mathscr{C}$. Hence, the proof is complete.

QED

V.4. Coinsurance Contracts

In this final subsection we shall analyze more closely the endogenous incomplete contracts in the class of simple coinsurance problems. We characterize two features of incomplete computable contracts selected by sensible criteria in this class of problems. First, we give conditions on the "accident state" $\bar{s}$ which are sufficient to guarantee that an endogenously incomplete contract will arise. Second, we are able to characterize the form that such incomplete contracts take. We find the intuition behind these features both "realistic" and appealing.

We start with an intuitive discussion of the sufficient conditions on $\bar{s}$. A recursively enumerable (r.e.) subset of **N** is a set of numbers that can be exhaustively "listed" by a Turing machine.[16] The existence of subsets of **N** that are *not* r.e. can easily be

16. Formally $\mathscr{E} \subseteq \mathbf{N}$ is r.e. if and only if there exists a Turing machine $M \in \mathbf{N}$ such that $\{M\}(n) \in \mathscr{E}$ for every n and for every $e \in \mathscr{E}$ there exists a natural number $n \in \mathbf{N}$ such that $\{M\}(n) = e$. (See Cutland [1980, Ch. 7] for further details.)

established by "counting arguments" and is standard in the literature on computability [Cutland 1980, Ch. 7].

Consider the binary expansion of a state of nature $s \in \mathscr{S}$ and the associated sets of "characteristics" that are present or not in s. Formally, let

$$\mathscr{K}_0(s) \equiv \{n \in \mathbf{N} \mid s_n = 0\}, \qquad \mathscr{K}_1(s) \equiv \{n \in \mathbf{N} \mid s_n = 1\},$$

where, as in (4), s_n stands for the nth digit of the binary expansion of s. The class of accident states which we will identify as yielding incomplete coinsurance contracts are states $\bar{s}$ for which neither $\mathscr{K}_0(\bar{s})$ nor $\mathscr{K}_1(\bar{s})$ are r.e. sets. Intuitively, these are states for which it is impossible to find an algorithm that lists either the characteristics that are present or those that are not. We shall refer to such real numbers as *algorithmically undescribable* or simply *undescribable*. The following lemma claims that such numbers abound in the interval [0,1]. This is proved by completely standard arguments (see Cutland [1980, Ch. 7]); hence we state it without proof.

LEMMA 5. The set of undescribable real numbers has Lebesgue measure 1 in the interval [0,1].

A longer and longer list of the digits in the binary expansion of any number clearly yields successive approximations to such number. Therefore, the following result—which we also state without proof—does not come as a surprise.

LEMMA 6. If $s \in \mathscr{S}$ is undescribable, there does not exist a Turing machine $M \in \mathbf{N}$ such that if $e_n = \{M\}(n)$ for every n, the sequence $\{e_n\}_{n=0}^{\infty}$ is monotonic and $\lim_{n\to\infty} e_n = s$.

In words, undescribable numbers cannot be approximated, from above or from below, in a computable way.

Proposition 3 below asserts that in a simple coinsurance problem, if the accident state $\bar{s}$ is undescribable, an incomplete contract will arise, and it will take the following form:

$$\{x\}(s) = \begin{cases} l' & \text{if } s \in [0,a_C) \\ h' & \text{if } s \in [a_C,b_C) \\ l' & \text{if } s \in [b_C,1], \end{cases} \tag{19}$$

where $0 \leq a_C < b_C \leq 1, \bar{s} \in [a_C,b_C)$, and $h' < l'$. Intuitively, if some sensible criterion $C \in \mathscr{C}$ gives rise to a complete limit partition, one could use the sequence $\{x_{n,C}\}_{n=0}^{\infty}$ to approximate computably the accident state $\bar{s}$. If $\bar{s}$ is undescribable, we know that this is not possible.

We are now ready to state formally our last result, which is proved in the Appendix.

PROPOSITION 3. Consider a simple coinsurance problem where the accident state $\bar{s}$ is undescribable. Then for every sensible choice criterion $C \in \mathscr{C}$,

(i) the limit partition $\mathscr{P}_C$ is incomplete;
(ii) the limit partition takes the form $\{[a_C,b_C);[0,a_C) \cup [b_C,1]\}$ with $\bar{s} \in [a_C,b_C)$.

The fact that according to Proposition 3 above genuinely incomplete contracts arise whenever the accident state is undescribable raises the issue of whether an undescribable first best is an unrealistic benchmark for our analysis. We do not think so. In fact, the type of situation we have in mind is a situation in which the parties have a clear idea of what they would like to describe in the contract, but they cannot provide a finite description of the characteristics that exactly define it. The following quote may be seen as an example of this, arguably common, type of situation:

> Year before (in Jacobellis v. Ohio, 1964), [Potter] Stewart had written that only "hardcore" pornography could be banned, but conceded the subjective nature of any definition: "I shall not today attempt to further define the kind of materials I understand to be embraced within the shorthand definition; and perhaps I could never succeed in doing so," Stewart had said. "But I know it when I see it" [Woodward and Armstrong 1979, p. 194].

The characterization presented in Proposition 3 lends itself easily to an interpretation of the resulting incomplete contracts in terms of standardized contracts. In fact, standard insurance contracts, for example, define the accident state that entitles the insured party to a compensation through a finite number of conditions that need to be satisfied. We can think of these conditions as roughly describing the accident state that is undescribable. The resulting contract will look like the one characterized by Proposition 3 which does not use a full description of the relevant characteristics of the accident state, since a procedure that lists them cannot be constructed.

VI. CONCLUDING REMARKS

This paper explores the extent to which contracts' incompleteness can be attributed to the *algorithmic nature* of contracts. We show that the algorithmic nature of contracts by itself is not enough to obtain genuinely incomplete contracts. On the other hand, if a similar restriction is imposed on the choice process on the part of the contracting parties, *endogenously incomplete* optimal contracts are found. Further, in the case of a simple coinsurance agreement, the optimal incomplete contract parties write takes a very simple and realistic form.

The incompleteness of contracts in our analysis is driven by the fact that it may be impossible to describe accurately in a written (and hence algorithmic) contract some relevant states of nature. As in, for instance, Grossman and Hart [1986] and Hart and Moore [1990], the difficulty is entirely ex ante. Once the state of nature is realized, the description of it may be redundant, and hence all difficulties may disappear. Modeling explicitly what aspects of a contracting problem make it possible to contract tomorrow (after the state is realized) on something which is impossible to contract ex ante seems a very important direction for future research.

Our approach focuses on the limits that are imposed on contracts by the available technology for describing future states of nature and derives optimal incomplete contracts that describe in an approximate way the critical states of nature. An alternative cause for contract incompleteness can be envisaged [Grossman and Hart 1986] in the available technology for describing the other essential element of a contract: the parties' expected performance and the mapping from performance to remuneration. We explore the consequences of imposing computability constraints on the possible contracts in this type of contracting problem in Anderlini and Felli [1993b]. We find that incompleteness may be more pervasive than in the present context because natural discontinuities destroy the force of the approximation result of Section IV above.

APPENDIX

Definitions of a Computable Contract

The second of the two equivalent definitions of computable contracts we presented in subsection II.3 above, Definition 2, is that of a *two-tape Turing machine.*[17]

17. See Hopcroft and Ullman [1979, Ch. 7] for definitions of Turing machines with any number of tapes and various equivalence theorems. It should also be noted that the use we make of the precise specification of a two-tape Turing machine is completely inessential. Everything that follows could be done in the context of an appropriately "programmed" standard Turing machine. We choose the specification below because we believe it helps intuitive reasoning about the way computable contracts work.

One of the two tapes is a read-only (r.o.) tape, while the other is a read/write (r.w.) tape. Given a state $s \in \mathscr{S}$ the binary expansion of this real number is placed on the machine's r.o. tape. As mentioned in subsection II.3 above, we follow the convention to choose the binary expansion containing the largest number of *zero* digits. The machine then starts operating according to its program, using as input what is read from the r.o. tape and using the r.w. tape for whatever intermediate operations are necessary. When the machine eventually "halts" (which we assume it does given any $s \in \mathscr{S}$), whatever is left on the r.w. tape is taken to be the machine's output on input s. Since we assume that the machine halts in a finite number of steps, the actual computation will only ever involve "scanning" a finite number of digits on the r.o. tape.

The alternative Definition 1 of a computable contract defines it as a pair of computable functions, (f,g), where the function $f:\mathscr{S} \to \mathbf{N}$ is the *information-gathering procedure* and the function $g:\mathbf{N} \to \mathbf{N}$ is the *outcome procedure.*

Denote by $D(\mathscr{D};s)$ the set of digits of the binary representation of s in the set of positions $\mathscr{D}$, where $\mathscr{D}$ is a finite subset of the naturals $\mathbf{N}$ and s is the state of nature in its binary representation. For example, if $s = 010110\ldots$ and $\mathscr{D} = \{1;3;4;6\}$, then $D(\mathscr{D};s) = \{(1,0);(3,0);(4,1);(6,0)\}$. Note that for every set $\mathscr{D} \subset \mathbf{N}$ finite and every $s \in \mathscr{S}$ the set of pairs $D(\mathscr{D};s)$ can be given a Gödel number. Also all finite subsets of $\mathbf{N}$ can be given a Gödel number. Hence $D(\cdot;\cdot)$ can be thought of as a mapping: $D:\mathbf{N} \times \mathscr{S} \to \mathbf{N}$.

The information-gathering procedure is, then, defined as follows. At the initial stage the set of digits scanned by the information gathering procedure is empty: $D(\varnothing;s) = \varnothing$, for every s. The first step of the procedure consists of scanning some digit of the binary expansion of s, given that no digits have been scanned before:

$$\mathscr{D}_1 = \tilde{f}(\varnothing),$$

where $\mathscr{D}_1$ is the set of scanned digits in the first step and $\tilde{f}:\mathbf{N} \to \mathbf{N}$ is the scanning procedure that reads step by step the digits of the binary expansion of a real number. Hence, after the first step the set of pairs $D(\mathscr{D}_1;s)$ is the information available about s. Define recursively

$$\mathscr{D}_n = \tilde{f}[D(\mathscr{D}_{n-1};s)] \tag{A.1}$$

so that $D(\mathscr{D}_n; s)$ is the information available after n steps (including the one defined on the empty set). Assume that $\tilde{f}(\cdot)$ satisfies the following properties:

(D1) The scanning procedure $\tilde{f}[D(\mathscr{D}_n; s)]$ is defined for every state of nature s and every index $n \in \mathbf{N}$.

(D2) For every $s \in \mathscr{S}$ there exists an $n \in \mathbf{N}$ such that $\mathscr{D}_{n+1} = \mathscr{D}_n$ (and hence $\mathscr{D}_{n+i} = \mathscr{D}_n$, $\forall i \in \mathbf{N}$).

(D3) The scanning procedure $\tilde{f}$:$\mathbf{N} \rightarrow \mathbf{N}$ is *computable*.

Properties (D1) and (D2) require the procedure $\tilde{f}$ to be well defined for every feasible state of nature s and to stop after a finite number of rounds. Property (D3) simply stipulates that the procedure should be computable in the Turing sense.

Finally, define $\bar{n}(s)$ as *the least n* such that $\mathscr{D}_{n+1} = \mathscr{D}_n$, for a given state of nature s. We have now all the elements to define the information-gathering procedure $f(\cdot)$. For any state of nature s let

$$f(s) \equiv D(\mathscr{D}_{\bar{n}(s)}; s). \tag{A.2}$$

In words, $f(s)$ is the total information gathered if the state of nature is s.

The outcome procedure is simple to describe. Such procedure, g:$\mathbf{N} \rightarrow \mathbf{N}$, maps the output of the information-gathering procedure, $f(s) = D(\mathscr{D}_{\bar{n}(s)}; s)$, into a regular rational number $c \in \mathscr{Q}$ which is the value of the computable sharing rule (contract) for the given state of nature s. We require such outcome function $g(\cdot)$ to be defined for every $s \in \mathscr{S}$.

The composition of the procedures $f(\cdot)$ and $g(\cdot)$ delivers the value of a computable contract for every realized state of nature s:

$$c = g\{f[D(\mathscr{D}_{\bar{n}(s)}; s)]\}.$$

As we mentioned above, the two alternative definitions of a computable contract Definition 1 and 2 coincide. This result is stated and proved in the following lemma.

LEMMA A.1. Definitions 1 and 2 are equivalent, in the sense that

- for every pair of functions (f,g) satisfying Definition 1 there exists a Turing machine $x \in \mathbf{N}$ satisfying Definition 2 and such that

$$\{x\}(s) = g\{f[D(\mathscr{D}_{\bar{n}(s)}; s)]\} \qquad \forall s \in \mathscr{S}; \tag{A.3}$$

- for every two-tape (r.w. and r.o.) Turing machine $x \in \mathbf{N}$ satisfying Definition 2 there exists a pair of functions, (f,g), satisfying Definition 1 and equation (A.3).

Proof of Lemma A.1. All the arguments involved are standard in the computability literature, hence we only sketch the argument. We proceed by construction. Given any pair of functions (f,g) satisfying Definition 1, construct a Turing machine $x \in \mathbf{N}$ which, for any state of nature s, first computes $f(s)$, step by step, scanning the r.o. tape and transcribing the final outcome on the r.w. tape. Once this is done, $\{x\}(s)$ computes $g(f(s))$ using the r.w. tape only, since g is, by definition, computable. The resulting $\{x\}(\cdot)$ satisfies, by construction, Definition 2 and equation (A.3).

Given any Turing maching $\{x\}(\cdot)$ satisfying Definition 2, construct a pair (f,g) as follows. For any state of nature s, let the function f simulate the whole computation of $\{x\}(s)$ and output as $f(s)$ the set of squares of the r.o. tape scanned by $\{x\}(s)$ during the computation. Note that the output of $\{x\}(s)$ must be a computable function of the set of squares of the r.o. tape scanned by $\{x\}(s)$ during the computation. Hence, g can just be set equal to such a computable function. The resulting pair (f,g) satisfies, by construction, Definition 1 and equation (A.3).

QED

A computable contract as defined above can always be associated with a natural number $x \in \mathbf{N}$. This can equivalently be taken to be either the Gödel number of the Turing machine of Definition 2 or the Gödel number of the pair of computable functions in Definition 1.

Proof of Lemma 1

Property (i) follows from the definition of $\Phi(\{x\};s)$, $\phi(\{x\},s)$ and equation (A.3) from which $s' \in \Phi(\{x\};s)$ implies that $f(s') = f(s)$. Property (ii) follows from the assumption that $f(s)$ is well defined for every state of nature $s \in \mathscr{S}$. Finally, we prove property (iii) by contradiction. Assume that the statement (iii) is false. Hence, we should be able to find s' and s'' such that $\Phi(\{x\};s') \cap \Phi(\{x\};s'') \neq \varnothing$ and $\Phi(\{x\};s') \neq \Phi(\{x\};s'')$. Note that $\Phi(\{x\};s') \neq \Phi(\{x\};s'')$ implies that $s' \neq s''$. Moreover, the definition of $\Phi(\cdot;\cdot)$ and $\Phi(\{x\};s') \cap \Phi(\{x\};s'') \neq \varnothing$ imply that the first $\hat{n} = \min\{n(\{x\},s'),n(\{x\},s'')\}$ digits of $\phi(\{x\},s')$ and $\phi(\{x\},s'')$ have to coincide. Hence, since by definition of $\Phi(\cdot;\cdot)$ and $\Phi(\{x\};s') \neq \Phi(\{x\};s'')$, it must be that $n(\{x\},s') \neq n(\{x\},s'')$, without any loss in generality we assume that $n(\{x\},s') < n(\{x\},s'')$. Define $D(n,\{x\},s)$ as the nth set of digits scanned by $\{x\}(s)$ on state s. (In terms of previously introduced notation $D(n,\{x\},s) = \mathscr{D}_n$ as defined in equation (A.1), the only difference being in the dependence from $\{x\}(\cdot)$ and s made explicit.) Recalling the definition of $\bar{n}(s)$—as the least n such that $D(n + 1,\{x\},s) = D(n,\{x\},s)$—the coincidence of the first $n(\{x\},s')$ digits of the binary expansion of $\phi(\{x\},s')$ and $\phi(\{x\},s'')$ implies that

$$D(\bar{n}(s'),\{x\},s') = D(\bar{n}(s'),\{x\},s'')$$

and correspondingly,

$$D(\bar{n}(s') + i,\{x\},s') = D(\bar{n}(s') + i,\{x\},s'')$$

for all $i \in \mathbf{N}$. In other words, since the two states s' and s'' have the first $n(\{x\},s')$ digits of their binary expansion in common, the scanning of s'' has to stop after $\bar{n}(s')$ digits as well. This is a *contradiction* of $n(\{x\},s') < n(\{x\},s'')$.

QED

Proof of Proposition 1

Some more notation is needed first. Let

$$\text{(A.4)} \qquad U_i^* \equiv \int_{\mathcal{S}} U_i[x^*(s);s]\, d\mu(s) \qquad i = 1,2,$$

be the expected utility yielded to party i by the first-best sharing rule. Using standard results,[18] we can always decompose any probability measure μ on $\mathcal{S} = [0,1]$ into the weighted sum of two probability measures: a purely atomic measure α, and a nonatomic measure γ. We then denote $U_i^* = pU_i^{*\gamma} + (1 - p)U_i^{*\alpha}$, $i = 1,2$, where p is a real number between zero and one, and

$$\text{(A.5)} \quad U_i^{*\gamma} \equiv \int_{\mathcal{S}} U_i[x^*(s);s]d\gamma(s) \qquad U_i^{*\alpha} \equiv \sum_{n=1}^{\infty} U_i[x^*(s_n);s_n]\alpha(s_n)$$

with s_n, $(n \in \mathbf{N})$ denoting the atoms of α. Further, the symbol L_i will denote party i's maximum utility loss from a sharing rule taking the "wrong" values. In other words $L_i \equiv \sup_{s\in\mathcal{S};x\in\mathbf{R}} U_i(x;s) - \inf_{s\in\mathcal{S};x\in\mathbf{R}} U_i(x;s)$, which is bounded since we assumed $U_i(\cdot;\cdot)$ to be bounded from above and from below.

Fix now an arbitrary $\epsilon > 0$. Choose $\bar{n}$ sufficiently large so that

$$\text{(A.6)} \qquad L_i \sum_{n=\bar{n}}^{\infty} \alpha(s_n) < \frac{\epsilon}{2} \qquad i = 1,2.$$

18. See, for instance, Parthasarathy [1977], Remark 26.8.

Construct now a finite collection of half-open intervals around the first $\bar{n}$ atoms of α, $[\underline{g}_n, \bar{g}_n)$, $n = 1, \ldots, \bar{n}$ with the following properties. First, $s_n \in [\underline{g}_n; \bar{g}_n)$, for every $n = 1, \ldots, \bar{n}$. Second, let

(A.7) $$A = \bigcup_{n=1}^{\bar{n}} [\underline{g}_n, \bar{g}_n),$$

and ensure that

(A.8) $$L_i \gamma(A) < \frac{\epsilon}{2} \qquad i = 1,2.$$

Notice that the intervals $[\underline{g}_n, \bar{g}_n)$, $n = 1, \ldots, \bar{n}$ can always be chosen so that (A.8) holds since γ is nonatomic. Third, ensure that $\underline{g}_n$ and $\bar{g}_n$ are regular rational numbers. This is clearly always possible since such numbers are dense in $\mathscr{S}$.

Since $U_i(\cdot;\cdot)$ is continuous in its first argument, we can now choose a set of regular rational numbers $v_1; \ldots; v_{\bar{n}}$ such that

(A.9) $$|U_i(x^*(s_n);s_n) - U_i(v_n;s_n)| < \frac{\epsilon}{2} \quad \forall n = 1, \ldots, \bar{n} \quad i = 1,2.$$

Notice now that by Assumption 1 the integrals in (A.4) and hence the integrals in (A.5) are well defined. From the definition of a Lebesgue integral, the continuity of $U_i(\cdot;\cdot)$ in its first argument, and the fact that γ is nonatomic, we can then deduce the following. There exist two finite sets of regular rational numbers, $q_m \in \mathscr{S}$, $m = 0, \ldots, \bar{m}$ (with $q_0 = 0$ $q_{\bar{m}} = 1$), and $z_1, \ldots, z_{\bar{m}}$, such that if one defines the step function,

$$g(s) = z_m \Leftrightarrow s \in [q_{m-1}, q_m), \qquad m = 1, \ldots, \bar{m},$$

then

(A.10) $$\left| U_i^{*\gamma} - \int_{\mathscr{S}} U_i(g(s);s)d\gamma(s) \right| < \frac{\epsilon}{2}.$$

We now proceed to combine the intervals around the atoms constructed earlier with the step function $g(\cdot)$ to obtain the desired approximate computable contract. Consider the step function defined by

$$f(s) = \begin{cases} v_n & \text{if } s \in [\underline{g}_n, \bar{g}_n) \\ z_m & \text{if } s \in [q_{m-1}, q_m) \cap \bar{A}, \end{cases}$$

where $\bar{A}$ is the complement of A as defined in (A.7). Clearly, $f(\cdot)$ is a step function taking a finite set of regular rational values that partitions $\mathscr{S}$ into a finite set of half-open intervals which have as endpoints regular rational numbers. Therefore, by Lemma 2 there exists a computable contract $x \in \mathbf{N}$ such that $\{x\}(s) = f(s)$ for every $s \in \mathscr{S}$. It remains to show that the approximation property of the statement of the proposition holds for a sharing rule equal to $f(\cdot)$. Using (A.6), (A.8), (A.9), and (A.10), simple algebra shows that by construction we have

$$\left| U_i^{*\gamma} - \int_{\mathscr{S} \cup \bar{A}} U_i(f(s);s)d\gamma(s) \right| < \frac{\epsilon}{2} + L_i\gamma(A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad i = 1,2$$

and

$$\left| U_i^{*\alpha} - \int_{\mathscr{S} \cup A} U_i(f(s);s)d\alpha(s) \right| < L_i \sum_{s=\bar{s}}^{\infty} \alpha(s) + \frac{\epsilon}{2} \sum_{s=1}^{\bar{s}} \alpha(s) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad i = 1,2$$

and therefore, for any $0 \leq p \leq 1$,

$$\text{(A.11)} \quad \left| pU_i^{*\gamma} + (1-p)U_i^{*\alpha} - \int_{\mathscr{S} \cup \bar{A}} pU_i(f(s);s)d\gamma(s) - \int_{\mathscr{S} \cup A} (1-p)U_i(f(s);s)d\alpha(s) \right| < \epsilon \qquad i = 1,2.$$

Since $\mu = p\gamma + (1-p)\alpha$ and $U_i^* = pU_i^{*\gamma} + (1-p)U_i^{*\alpha}$, from (A.11) it follows that

$$\text{(A.12)} \qquad \left| U_i^* - \int_{\mathscr{S}} U_i[f(s);s]d\mu(s) \right| < \epsilon.$$

To conclude the argument, notice that by Assumption 2, for ϵ sufficiently small, (A.12) obviously implies (11). By Assumption 3, for ξ sufficiently small, the sharing rule $f(s)$ is feasible and hence the proof is complete.

QED

Proof of Proposition 3

If (i) holds, it is clear from the proof of Proposition 2 that (ii) follows. Hence, we only prove (i). Given a sensible criterion $C \in \mathscr{C}$, it is clear from the way we have defined the choice sequence associated with it that there exists a Turing machine $M_C \in \mathbf{N}$ such that $\{M_C\}(n) = x_{n,C}$ for every $n \in \mathbf{N}$. From the proof of Proposition 2 we know that for any sensible criterion $C \in \mathscr{C}$ there exists $\bar{n}$ such that $n \geq \bar{n}$ guarantees that

$$P(x_{n,C}) = \{[a(\bar{s};x_{n,C}),b(\bar{s};x_{n,C}));[0,a(\bar{s};x_{n,C})) \cup [b(\bar{s},x_{n,C}),1]\}$$

and moreover that if $\mathscr{P}_C$ is complete,

$$\lim_{n\to\infty} a(\bar{s};x_{n,C}) = \lim_{n\to\infty} b(\bar{s};x_{n,C}) = \bar{s}.$$

Let α_0 be a regular rational number in $[0,a[\bar{s},x_{\bar{n},C}))$. Assume now that (i) is false, and consider $C \in \mathscr{C}$ such that $\mathscr{P}_C$ is complete with $\bar{s}$ undescribable. In the rest of the proof we construct a Turing machine T which approximates computably $\bar{s}$ from below. This yields a *contradiction.*

Consider the following "steps" defining operations of Turing machine $T \in \mathbf{N}$ on input $m \in \mathbf{N}$.[19]

1) Compute $\{M_C\}(\bar{n} + z) = x_{\bar{n}+z,C}$, (start with $z = 0$).
2) Compute $\{x_{\bar{n}+z,C}\}(1)$.
3) Compute $\{x_{\bar{n}+z,C}\}(a_w + (1 - a_w)/h)$, (start with $w = 0$, $h = 1$).
4) Check whether $\{x_{\bar{n}+z,C}\}(a_w + (1 - a_w)/h) \neq \{x_{\bar{n}+z,C}\}(1)$.
5) If the answer to 4) is *no,* set $h: = h + 1$ and go to 3).
6) If the answer to 4) is *yes,* set $k_w = a_w + (1 - a_w)/h$.
7) Compute $\{x_{\bar{n}+z,C}\}(a_w + (k_w - a_w)/t)$ (start with $t = 1$).
8) Check whether $\{x_{\bar{n}+z,C}\}(a_w + (k_w - a_w)/t) = \{x_{\bar{n}+z,C}\}(0)$.
9) If the answer to 8) is *no,* set $t: = t + 1$ and go to 7).
10) If the answer to 8) is *yes,* set $a_{w+1} = a_w + (k_w - a_w)/t$.
11) Set $w: = w + 1$.
12) Check whether $w = m$. If *yes,* print a_w and *stop*; if *no,* set $z: = z + 1$.
13) Check whether $\{x_{\bar{n}+z,C}\}(a_w) = \{x_{\bar{n}+z,C}\}(0)$.
14) If the answer to 13) is *yes,* go to 1).
15) If the answer to 13) is *no,* set $z: = z + 1$ and go to 13).

It is clear from steps 1) to 15) that $m' > m$ implies that $\{T\}(m') > \{T\}(m)$, $\{T\}(m) < \bar{s}$ for every $m \in \mathbf{N}$ and

$$\lim_{m\to\infty} \{T\}(m) = \bar{s}.$$

This concludes the proof.

QED

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19. In the theory of computability it is widely accepted that if a "clear sequence of steps" can be defined performing a certain computation, then a Turing machine which performs the computation exists. Arguments of this type are known as *proof by Church's Thesis.* See, for instance, Cutland [1980, Ch. 1 and 3].

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